Low-overhead magic state distillation with color codes

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Based on **arXiv:2409.07707** with
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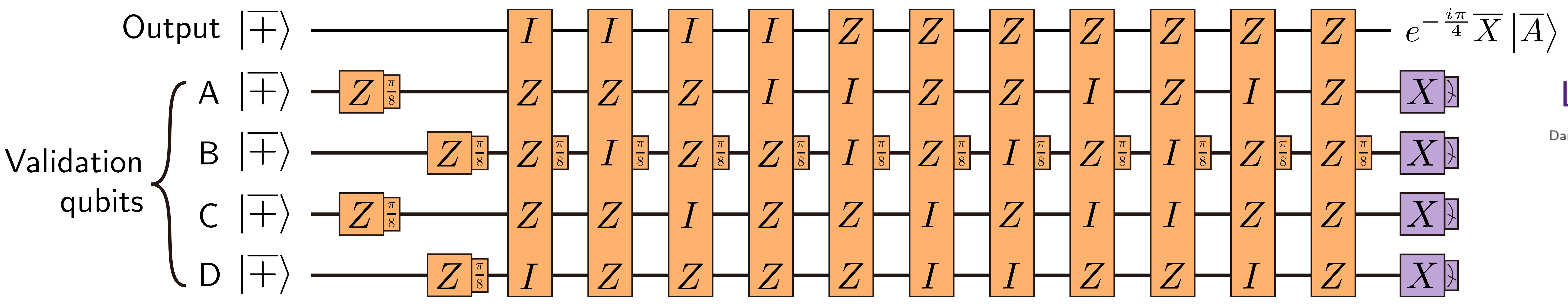
Motivation

- Magic state distillation (MSD) is **highly costly** due to its demand for many logical operations.
- We should **tailor & optimise** it to a specific code we want to use.
- Such optimisation has been studied well for **surface codes** by Litinski.
- How about for **color codes**?



A Game of Surface Codes: Large-Scale Quantum Computing with Lattice Surgery

Daniel Litinski @ Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

Litinski, Quantum 3, 128 (2019)

Magic State Distillation: Not as Costly as You Think

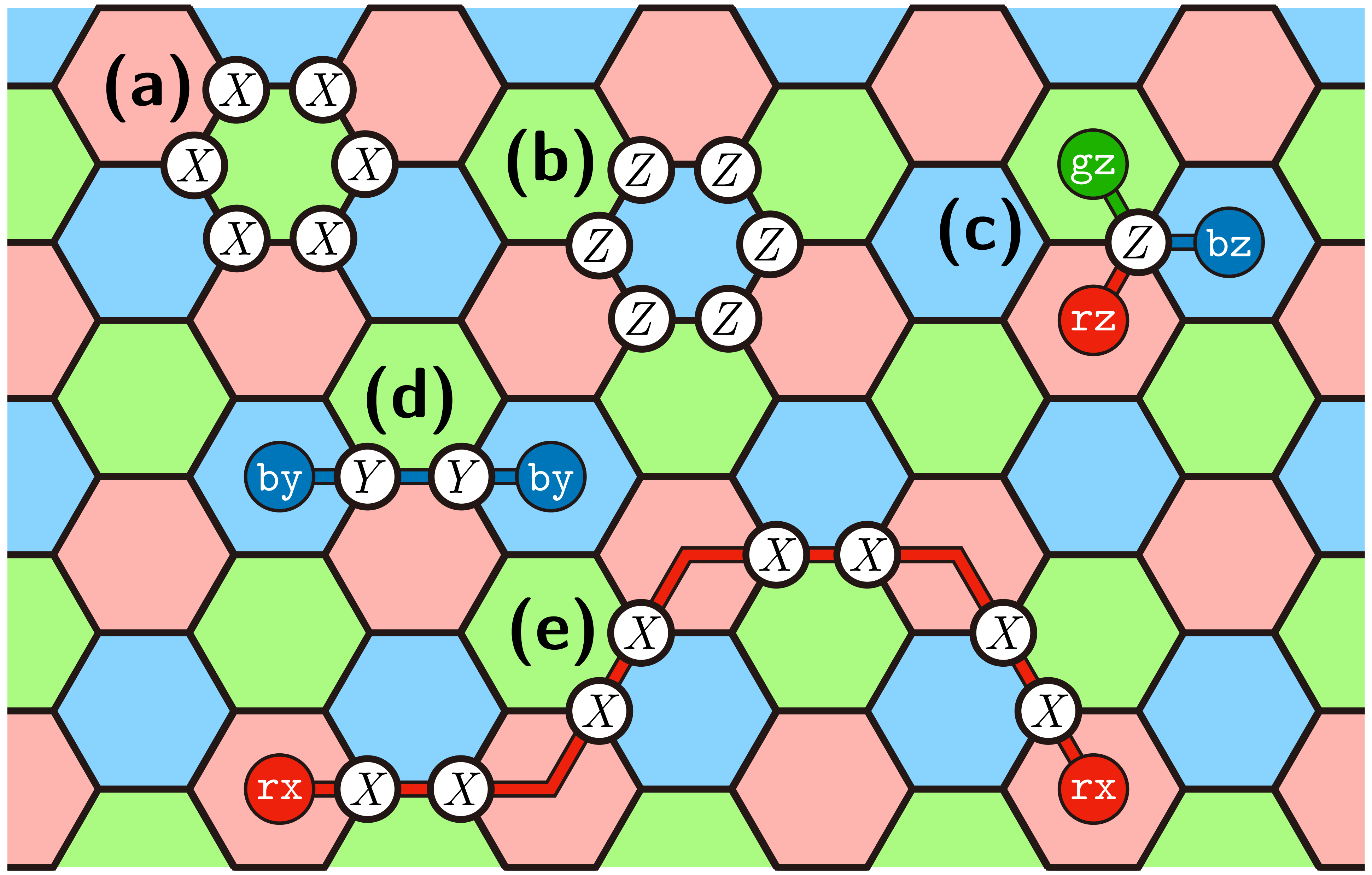
Daniel Litinski @ Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

Litinski, Quantum 3, 205 (2019).

Color Codes

Definition

- Color-code lattice [Bombin & Martin-Delgado, PRL 2006]
  - **3-valent**
  - **3-colorable** faces & edges
- Qubit on each vertex
- X-type and Z-type checks on each face



Color Codes

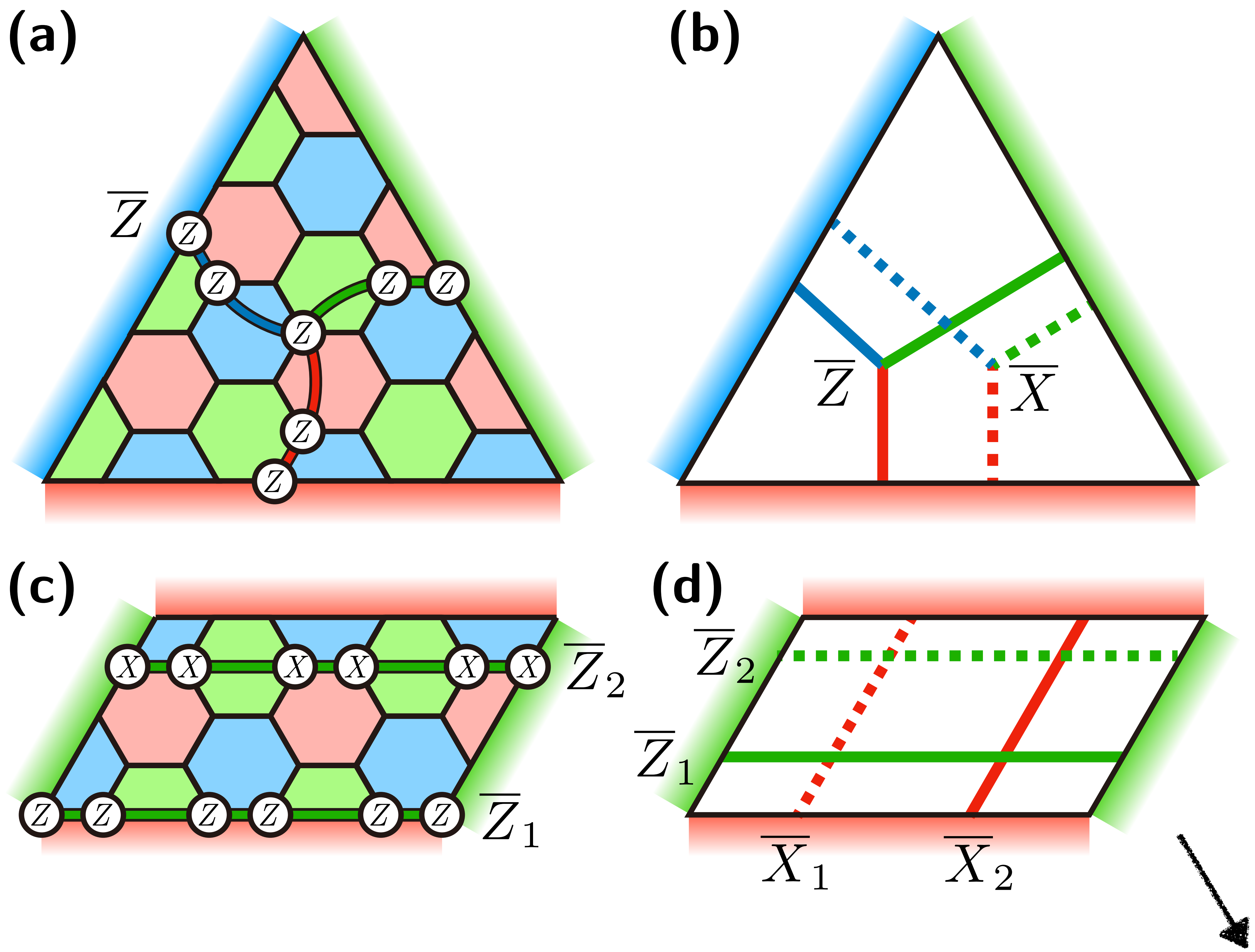
Logical Patches

Triangular patch

(single logical qubit)

Rectangular patch

(two logical qubits)



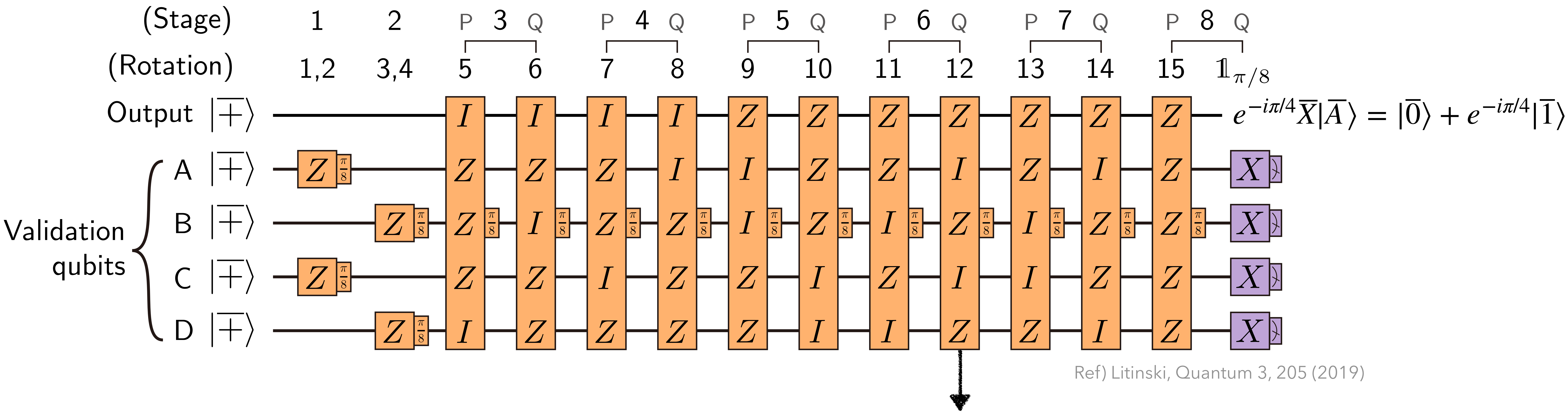
Solid lines: Pauli-Z strings

Dotted lines: Pauli-X strings

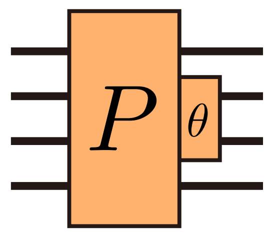
May have different code distances for logical X and Z errors.

Single-Level MSD Scheme

15-to-1 MSD circuit



Ref) Litinski, Quantum 3, 205 (2019)

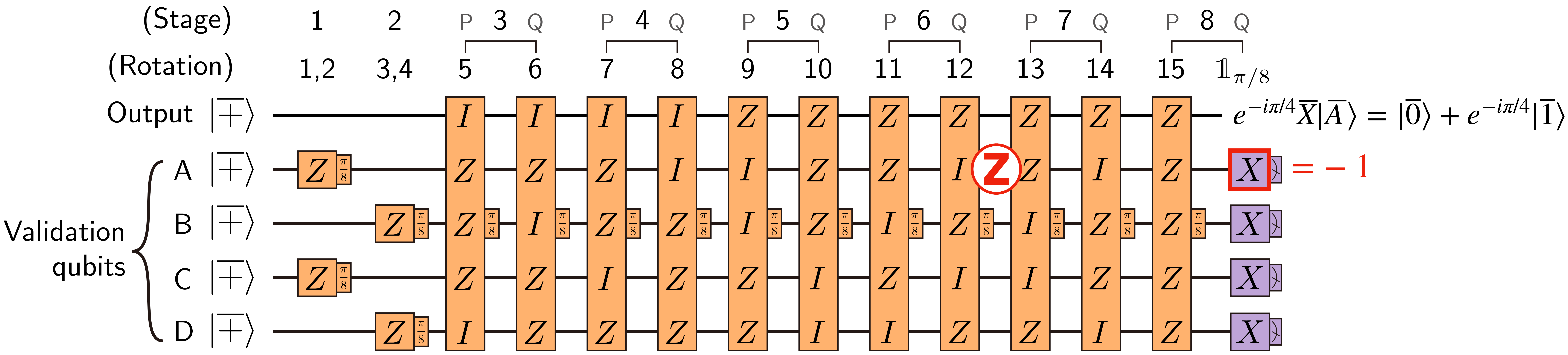
Faulty rotation gate $\overline{P}_{\pi/8}$  $= \overline{P}_\theta = e^{-i\theta\overline{P}}$

The circuit can tolerate

- **at most two rotation errors,**
- any $\overline{Z}$ errors on validation qubits,
- at most one $\overline{X}$ error on a validation qubit

Single-Level MSD Scheme

15-to-1 MSD circuit



Ref) Litinski, Quantum 3, 205 (2019)

The circuit can tolerate

- at most two rotation errors,
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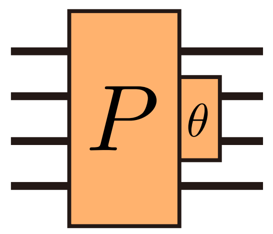
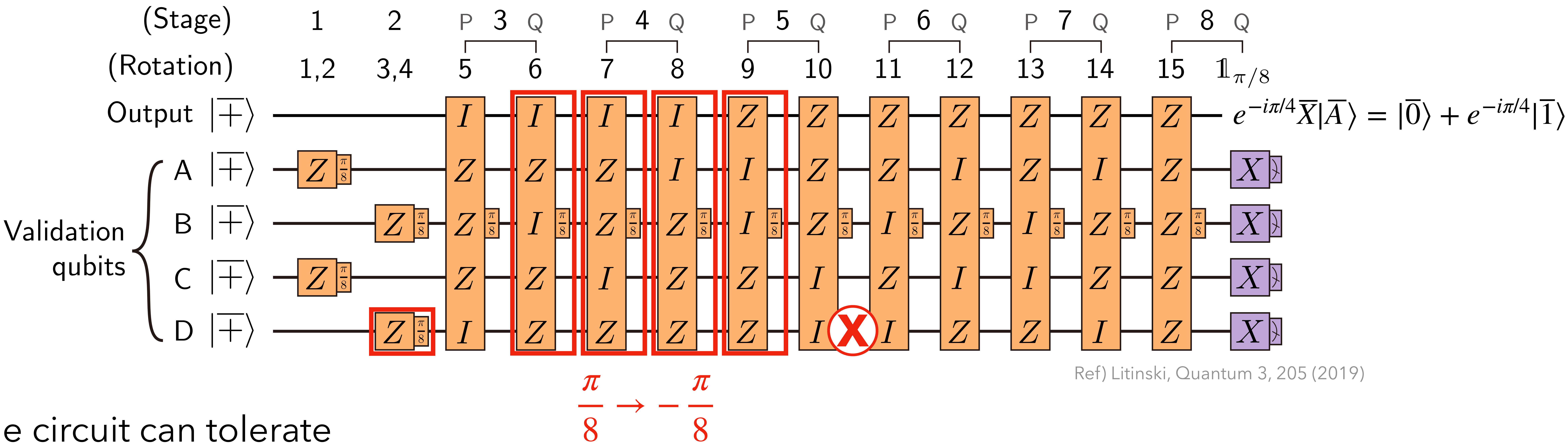


$= \overline{P}_\theta = e^{-i\theta\overline{P}}$

Single-Level MSD Scheme

15-to-1 MSD circuit



Ref) Litinski, Quantum 3, 205 (2019)

The circuit can tolerate

- at most two rotation errors,
- any $\overline{Z}$ errors on validation qubits,
- **at most one $\overline{X}$ error on a validation qubit**

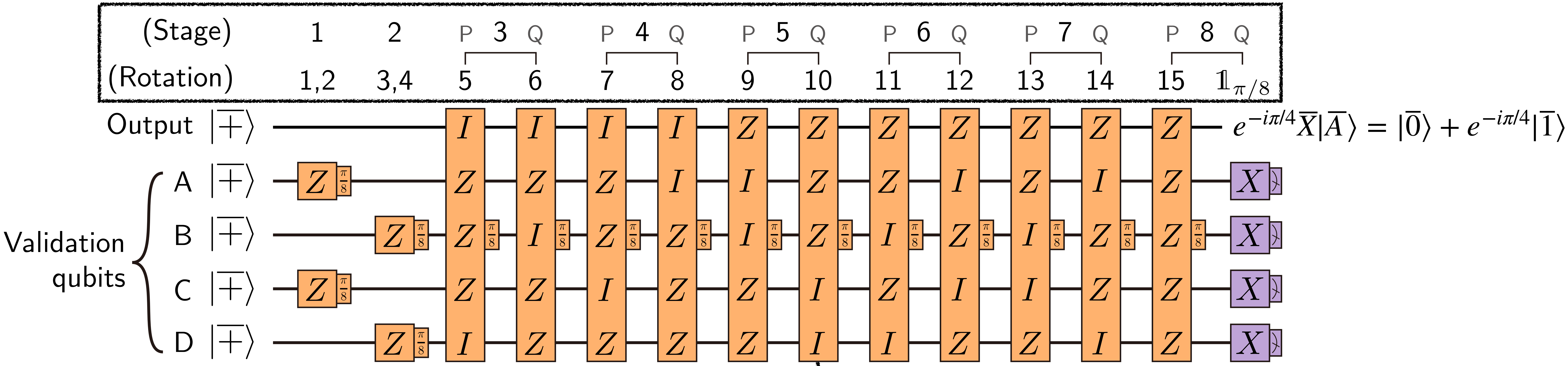
Does not damage the output qubit without being detected even though it causes more than two rotation errors

$$\prod_{P \in \mathscr{P}} P_{-\pi/4} \propto \prod_{P \in \mathscr{P}} (I + iP) = \sum_{\tilde{\mathscr{P}} \subseteq \mathscr{P}} \left[i^{|\tilde{\mathscr{P}}|} \prod_{P \in \tilde{\mathscr{P}}} P \right]$$

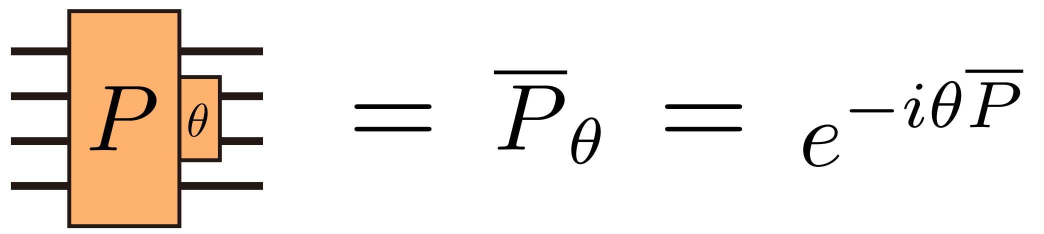
Single-Level MSD Scheme

15-to-1 MSD circuit

Configuration of rotation gates



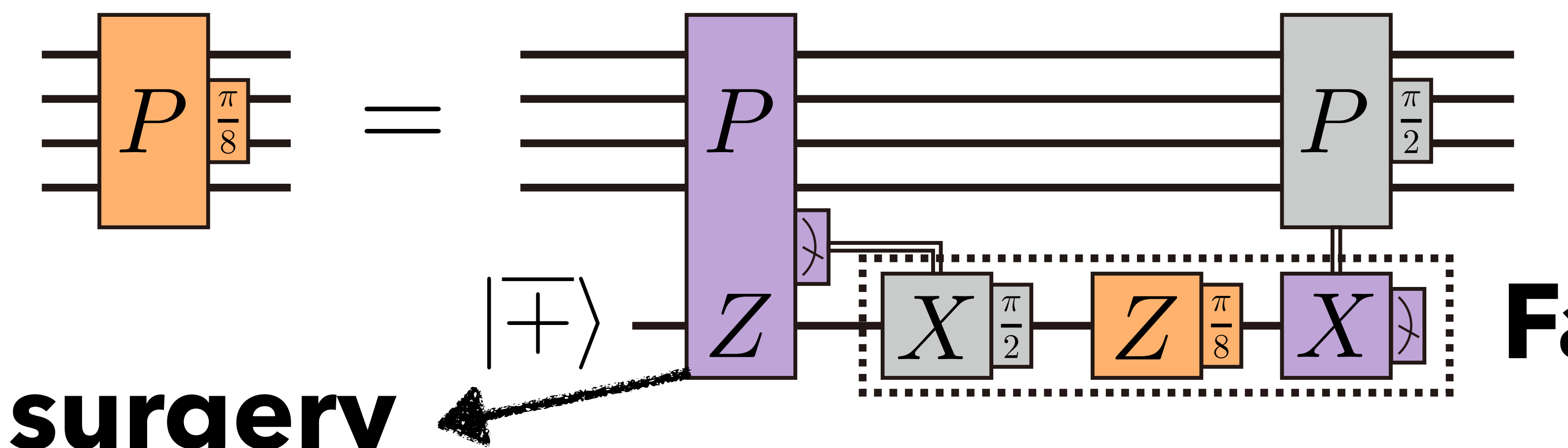
Ref) Litinski, Quantum 3, 205 (2019)



Five logical qubits

Layout

Faulty rotation gate $\overline{P}_{\pi/8}$

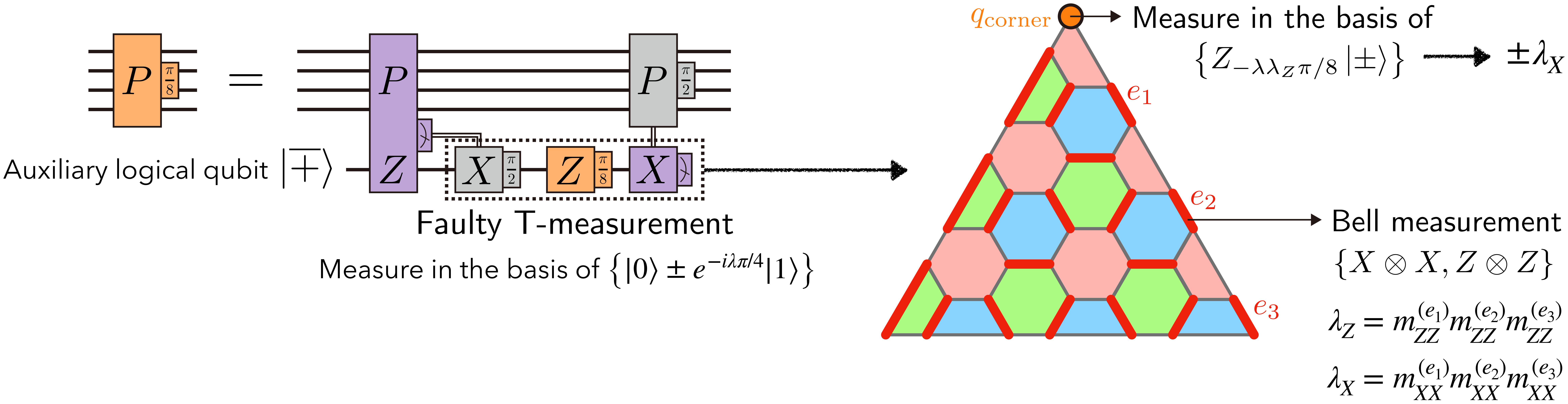


Faulty T-measurement

Lattice surgery

Single-Level MSD Scheme

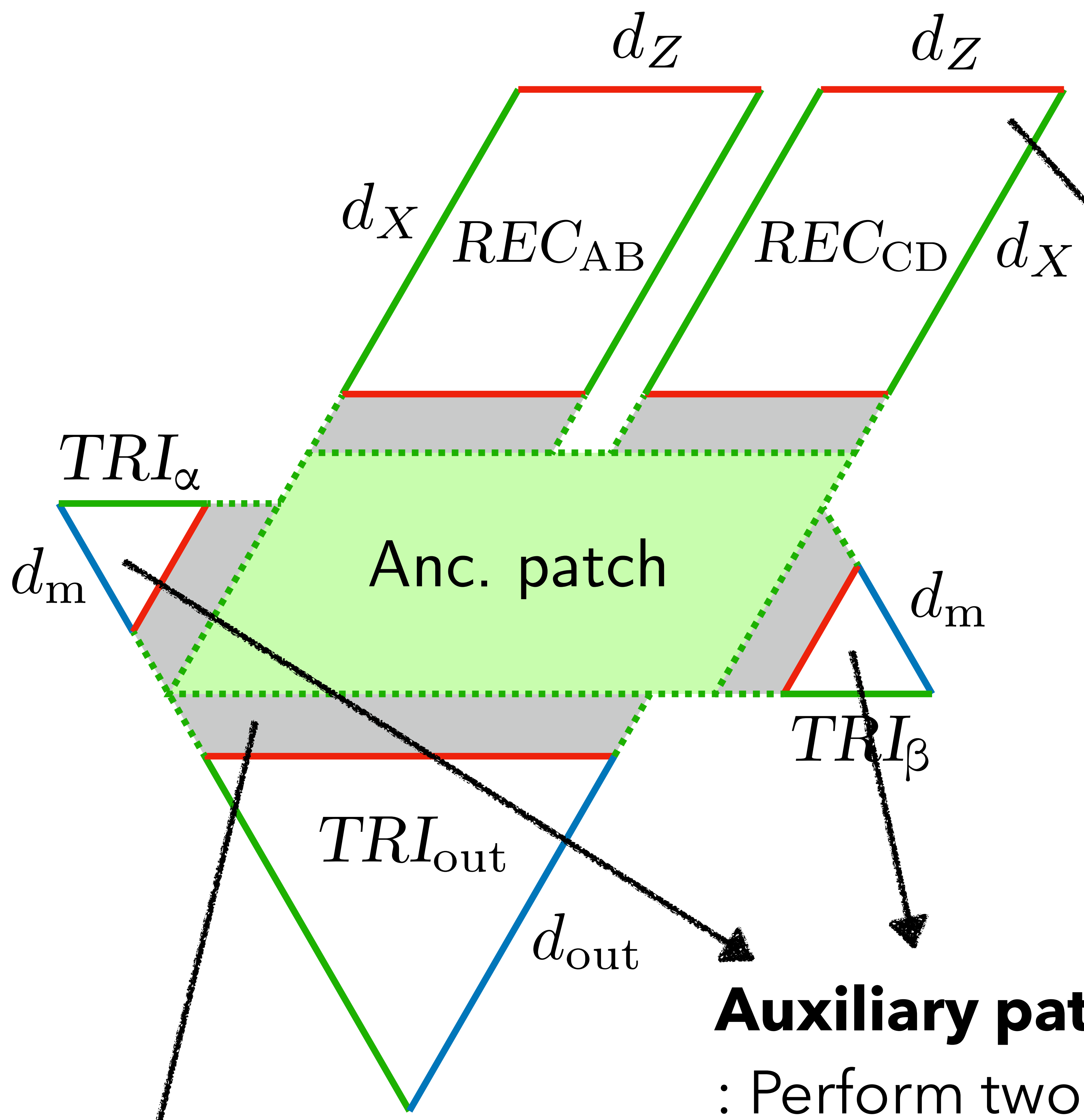
Faulty T-measurement

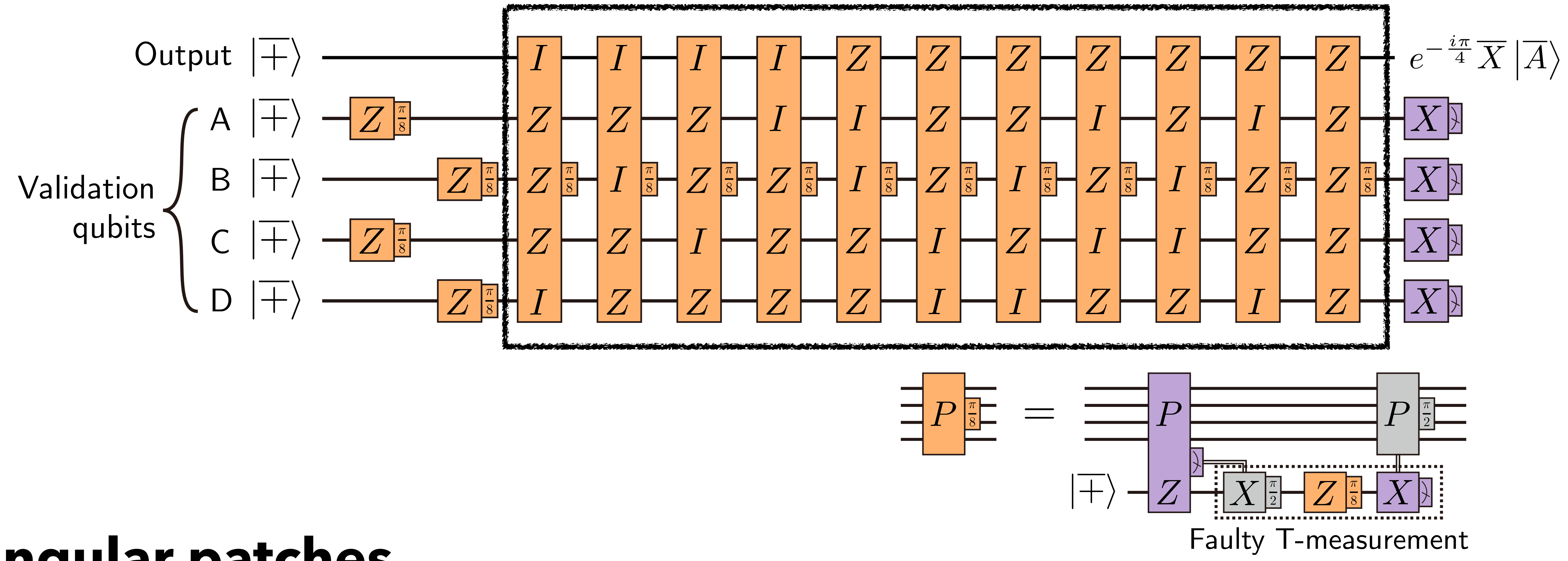


Single-Level MSD Scheme

Layout

sng-($d_{\mathrm{out}}, d_{\mathrm{X}}, d_{\mathrm{Z}}, d_{\mathrm{m}}$) scheme





Use rectangular patches

∵ (i) $\overline{X}$ and $\overline{Z}$ errors have different effects on the circuit and

(ii) only $\overline{Z}$ operators are involved in the lattice surgery.

Auxiliary patches for faulty T-measurement

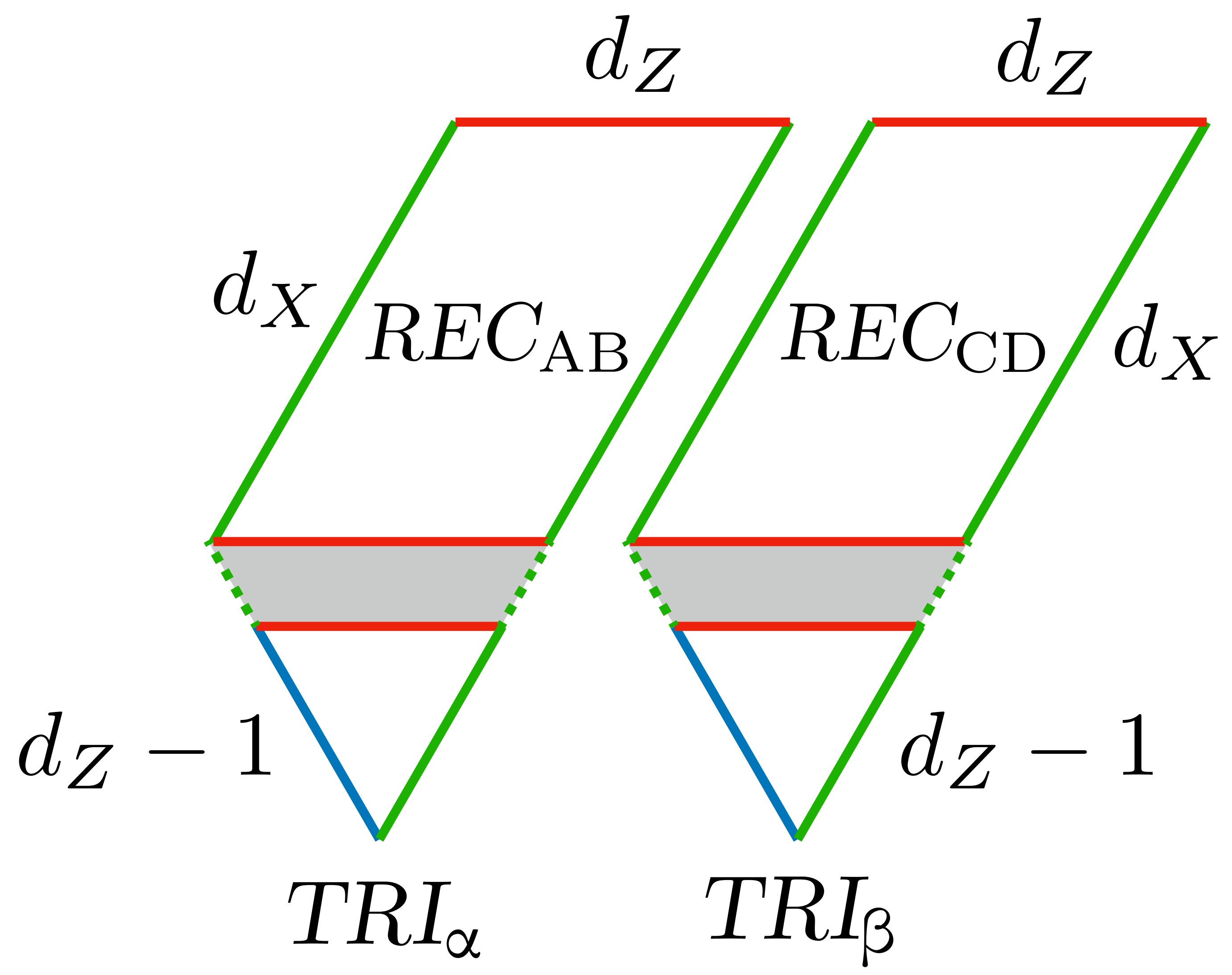
: Perform two faulty rotations at the same time.

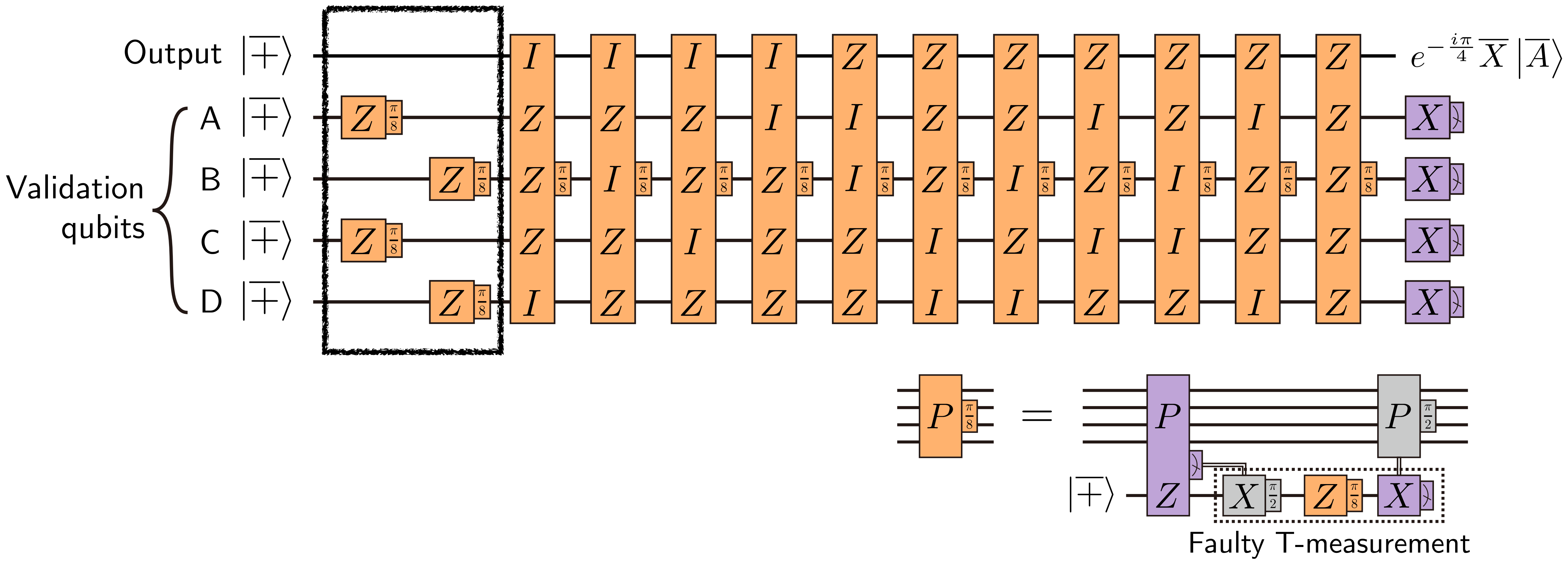
Interface regions for domain walls

Single-Level MSD Scheme

Layout

sng-(d_{out}, d_X, d_Z, d_m) scheme



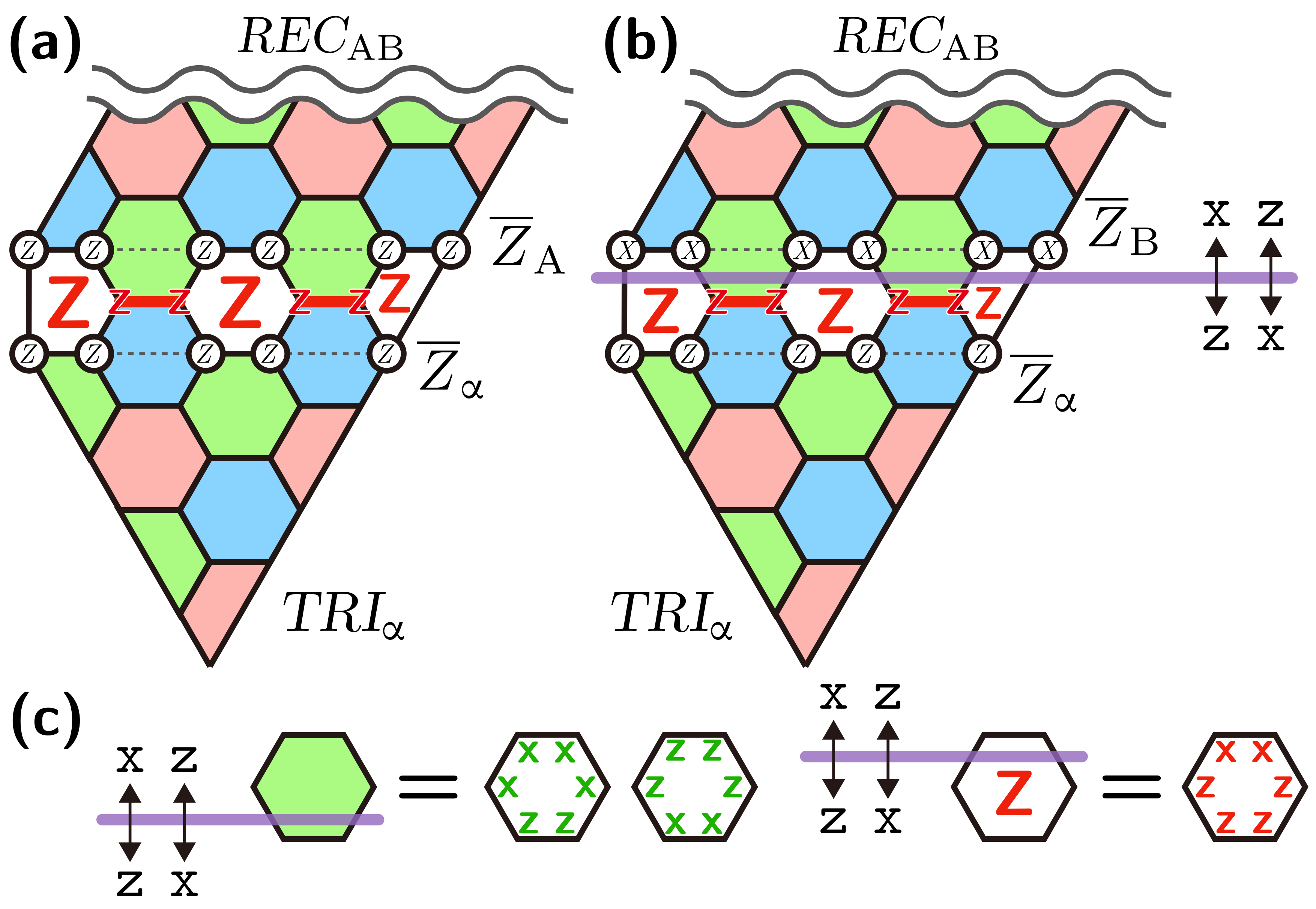


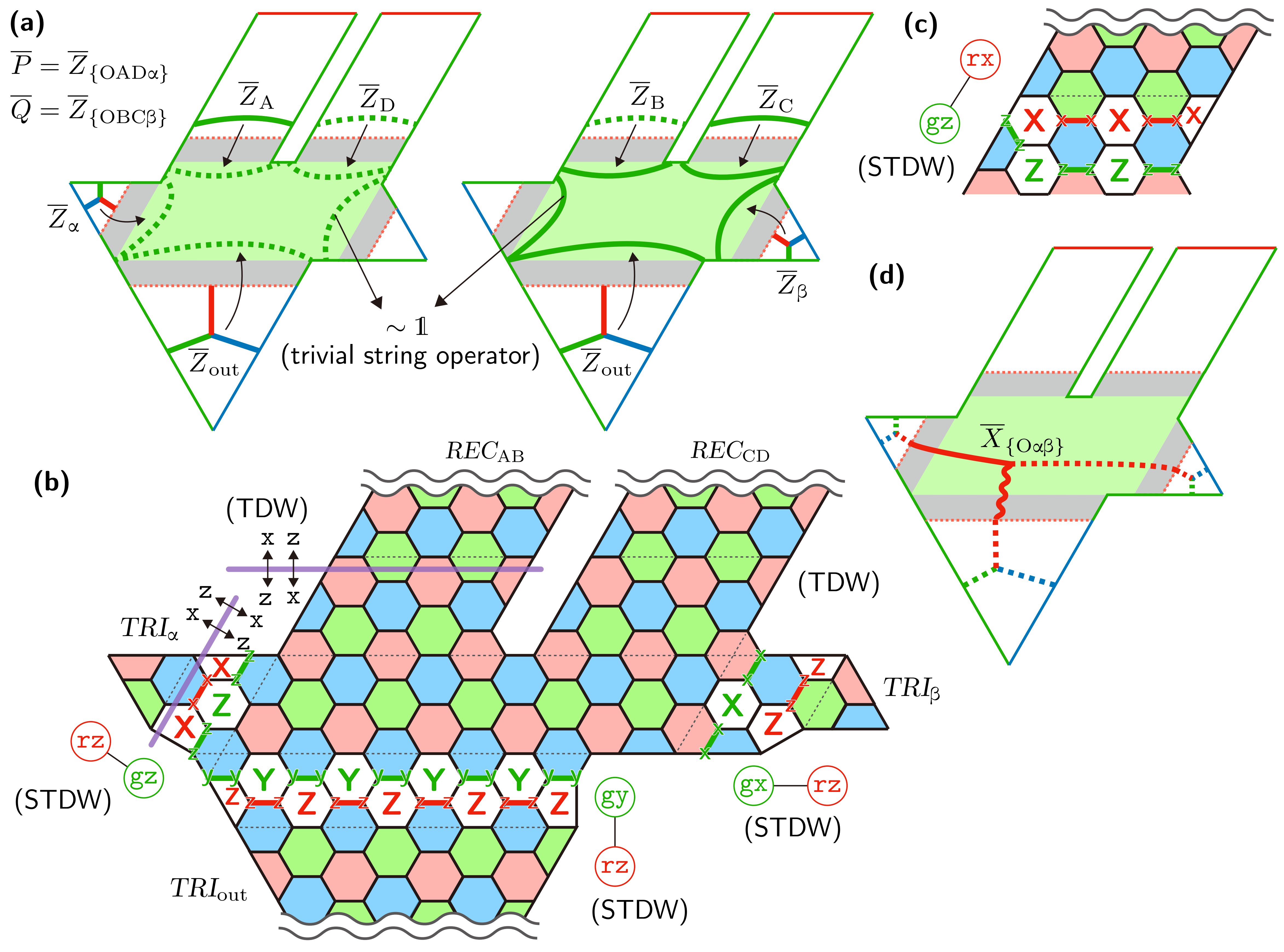
Single-Level MSD Scheme

Lattice Surgery

- Two commuting Pauli operators can be measured in parallel.

Ref) Thomsen et al., arXiv:2201.07806

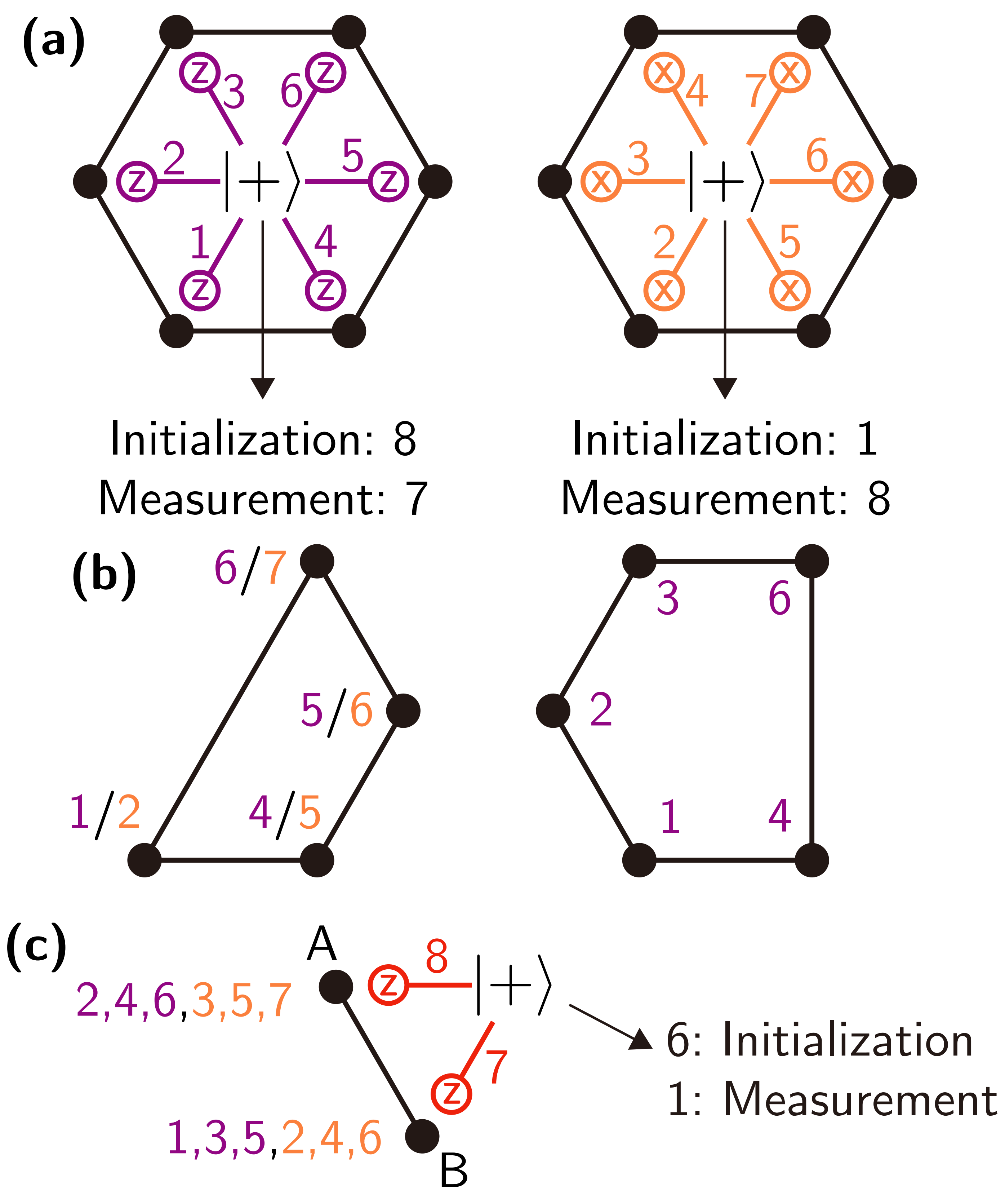




Single-Level MSD Scheme

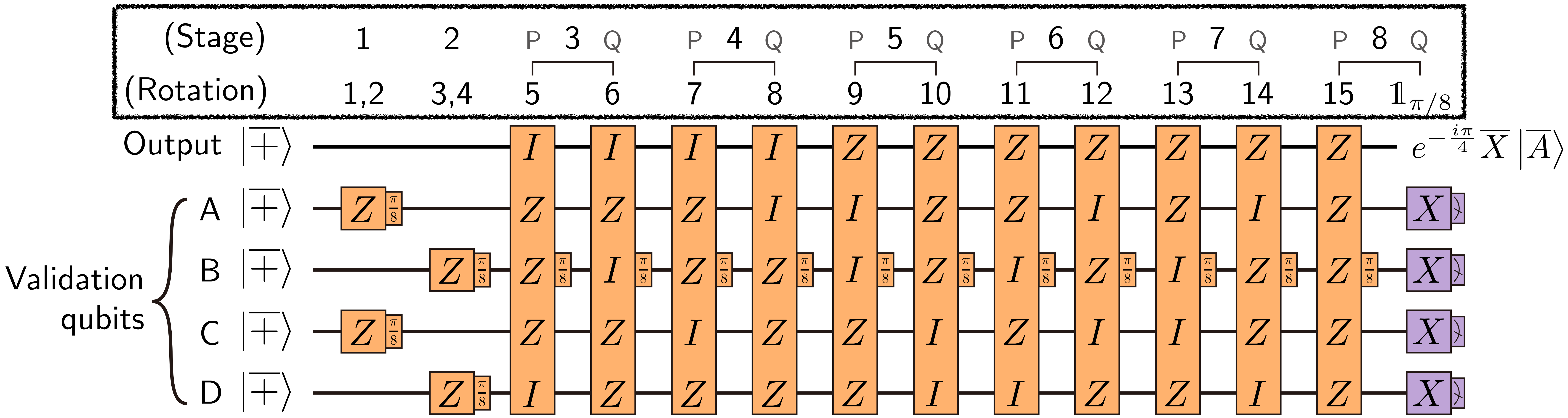
Syndrome Extraction Circuit

- Two-body check measurements can be done simultaneously with other check measurements.



Single-Level MSD Scheme

Configuration of Rotation Gates

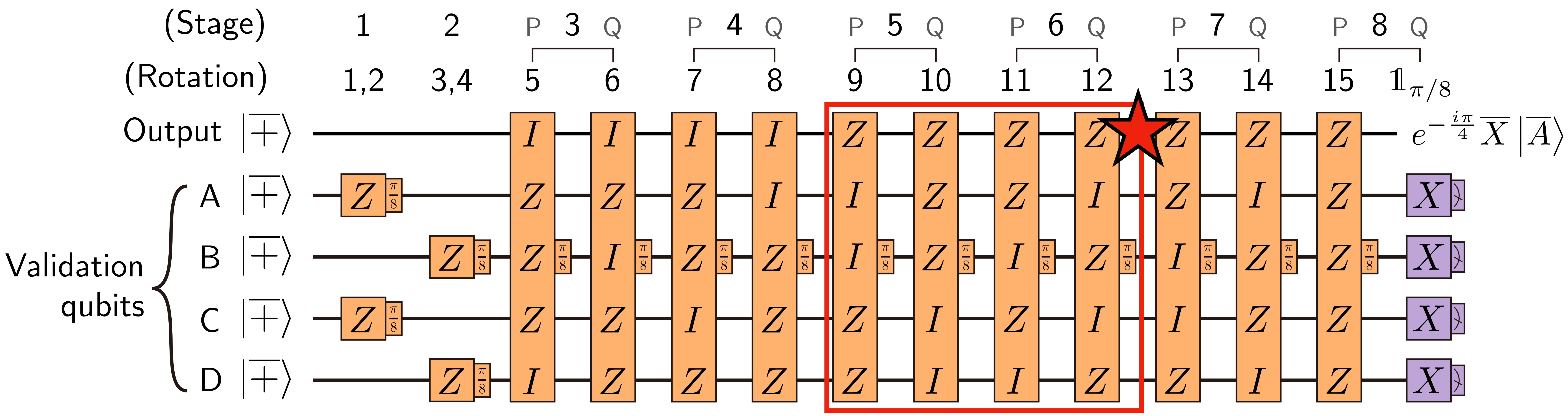


- Perform two rotations in each stage.
- We should consider two factors:

1. The **number of harmful errors** (that cause logical errors on the output state without being detected) **varies** depending on the configuration.
2. The layout is **not distance-preserving** for some configurations.

Single-Level MSD Scheme

Configuration of Rotation Gates



- Perform two rotations in each stage.
- We should consider two factors:

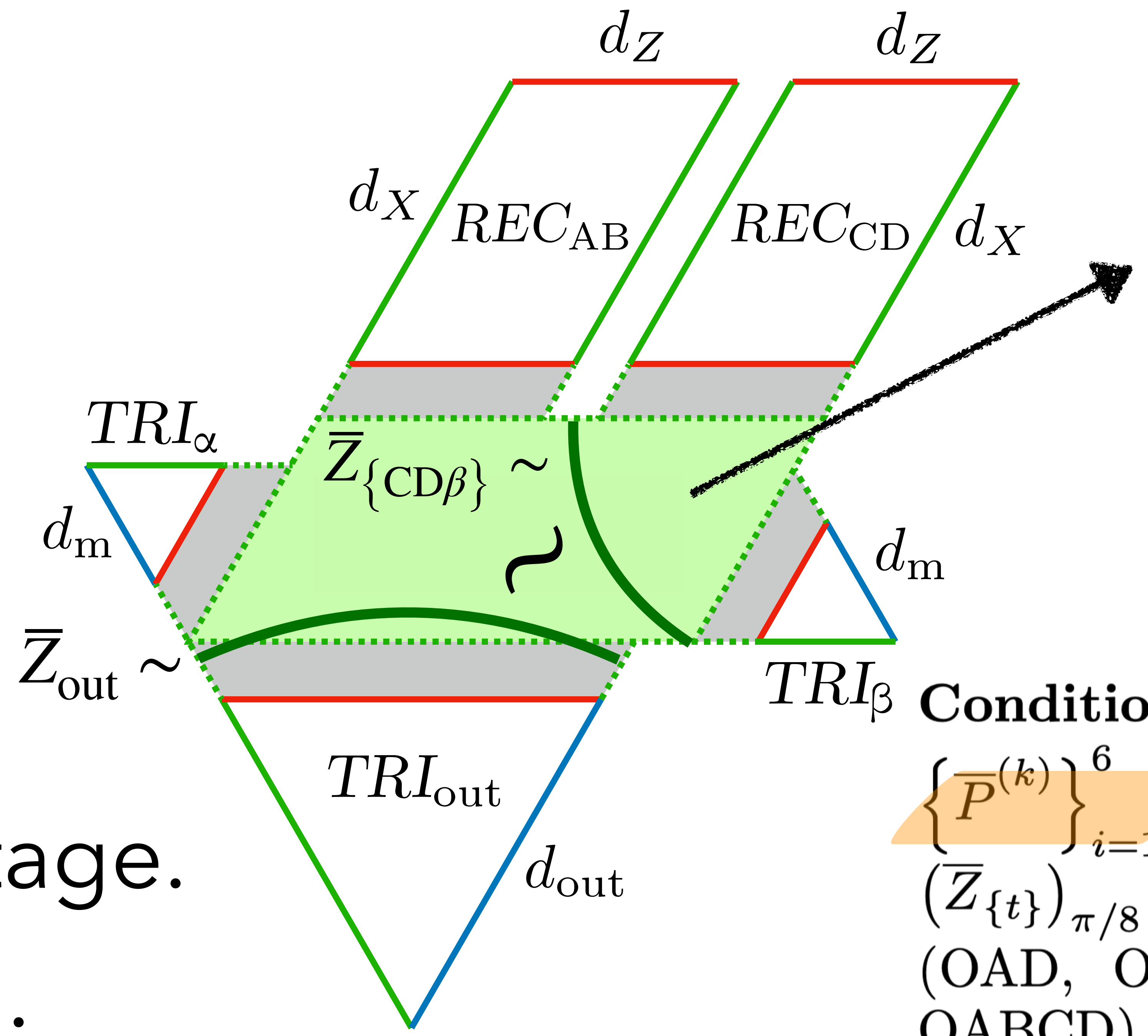
(i) Activate the output qubit as late as possible.

(ii) Make $\overline{X}_{\text{out}}$ errors right after stages 5 and 6 unharmful.

1. The **number of harmful errors** (that cause logical errors on the output state without being detected) **varies** depending on the configuration.
2. The layout is **not distance-preserving** for some configurations.

Single-Level MSD Scheme

Configuration of Rotation Gates



A $\overline{Z}_{\mathrm{out}}$ error can occur by error strings with weight $< d_{\mathrm{out}}$

No stages measure $\overline{Z}_{\{\mathrm{OCD}\beta\}}$

$\equiv$

Condition 1. (i) $d_{\mathrm{out}} - d_Z \leq d_{\mathrm{m}} < 2d_Z$, (ii) $\overline{Z}_{\{\mathrm{OCD}\}} \in \left\{P^{(k)}\right\}_{i=1}^{6}$, and (iii) if $d_Z > 2d_{\mathrm{m}} + 2$, $\left(\overline{Z}_{\{s\}}\right)_{\pi/8}$ and $\left(\overline{Z}_{\{t\}}\right)_{\pi/8}$ are not paired for each (s, t) in (OAC, OBC), (OAD, OBD), (OAC, OAD), (OBC, OBD), (OCD, OABCD), and (OAB, OABCD).

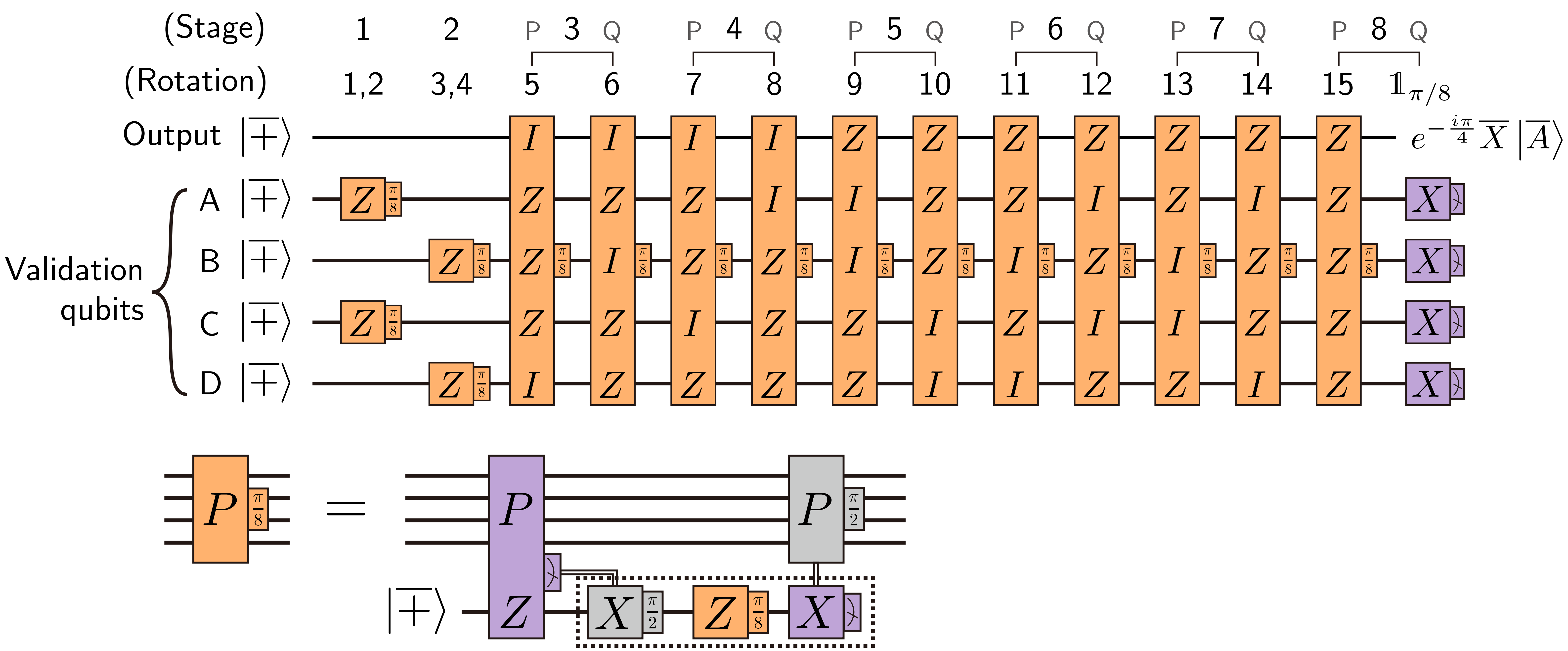
- Perform two rotations in each stage.
- We should consider two factors:

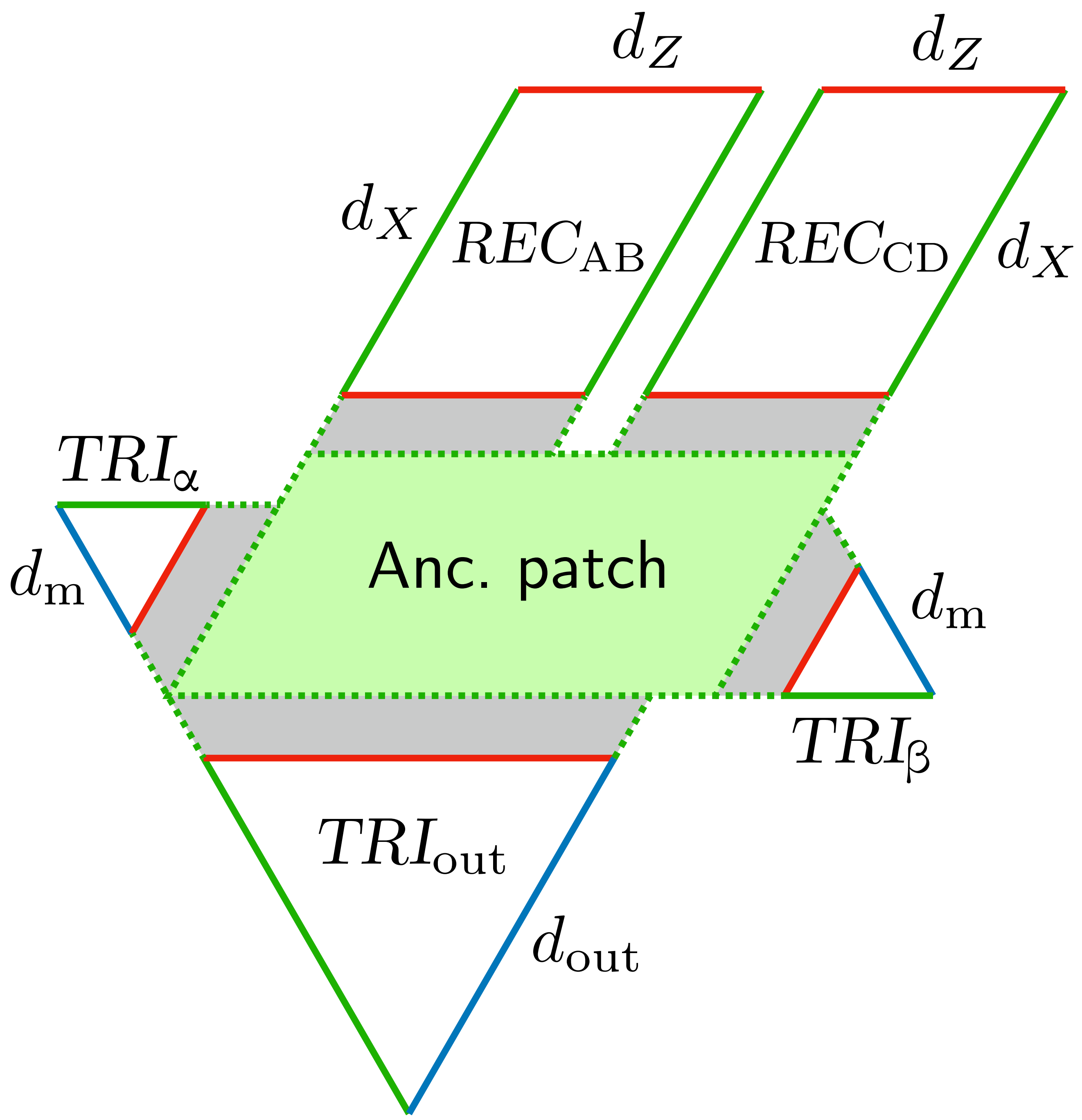
1. The **number of harmful errors** (that cause logical errors on the output state without being detected) **varies** depending on the configuration.
2. The layout is **not distance-preserving** for some configurations.

Single-Level MSD Scheme

Summary

- Basic ingredients: Lattice surgery & Faulty T-measurement
- By arranging the rotation gates properly,
  the number of harmful errors can be minimised & the layout can be distance-preserving.
- Output infidelity $\gtrapprox 35p^3$ for circuit-level physical error rate p





Production of Higher-Quality Magic States

Chamberland-Noh (CN) Protocol

- Distillation-free magic state preparation protocol
  - Fault-tolerantly prepare a H-type magic state $|\overline{H}\rangle := \cos\frac{\pi}{8}|\overline{0}\rangle + \sin\frac{\pi}{8}|\overline{1}\rangle = e^{i\pi/8}\overline{H}\overline{Z}_{\pi/4}|\overline{A}\rangle$ on a triangular color code with $d \le 7$
  - Using the transversality of the logical H gate
  - Non-destructively measure $\overline{H}$ through multiple controlled-H gates between data and ancilla qubits
  - Ensure fault-tolerance by flag qubits
  - Highly resource-efficient, but its output infidelity is limited (e.g., $\gtrapprox 10^{-10}$ for $p = 10^{-4}$)

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Very low overhead fault-tolerant magic state preparation using redundant ancilla encoding and flag qubits

Christopher Chamberland[1,2] and Kyungjoo Noh[1]

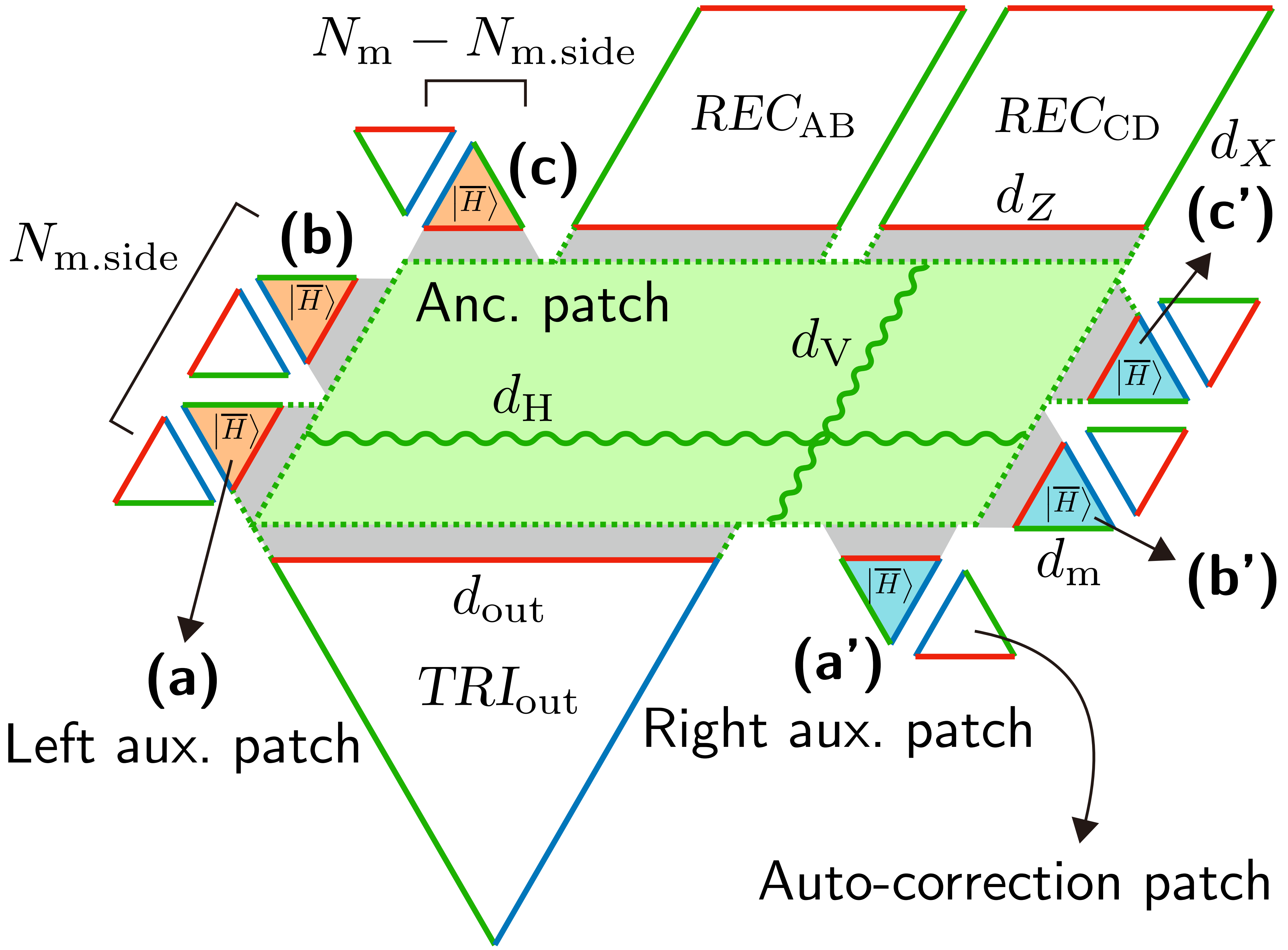
Fault-tolerant quantum computing promises significant computational speedup over classical computing for a variety of important problems. One of the biggest challenges for realizing fault-tolerant quantum computing is preparing magic states with sufficiently low error rates. Magic state distillation is one of the most efficient schemes for preparing high-quality magic states. However, since magic state distillation circuits are not fault-tolerant, all the operations in the distillation circuits must be encoded in a large distance error-correcting code, resulting in a significant resource overhead. Here, we propose a fault-tolerant scheme for directly preparing high-quality magic states, which makes magic state distillation unnecessary. In particular, we introduce a concept that we call redundant ancilla encoding. The latter combined with flag qubits allows for circuits to both measure stabilizer generators of some code, while also being able to measure global operators to fault-tolerantly prepare magic states, all using nearest neighbor interactions. We apply such schemes to a planar architecture of the triangular color code family and demonstrate that our scheme requires at least an order of magnitude fewer qubits and space–time overhead compared to the most competitive magic state distillation schemes. Since our scheme requires only nearest-neighbor interactions in a planar architecture, it is suitable for various quantum computing platforms currently under development.

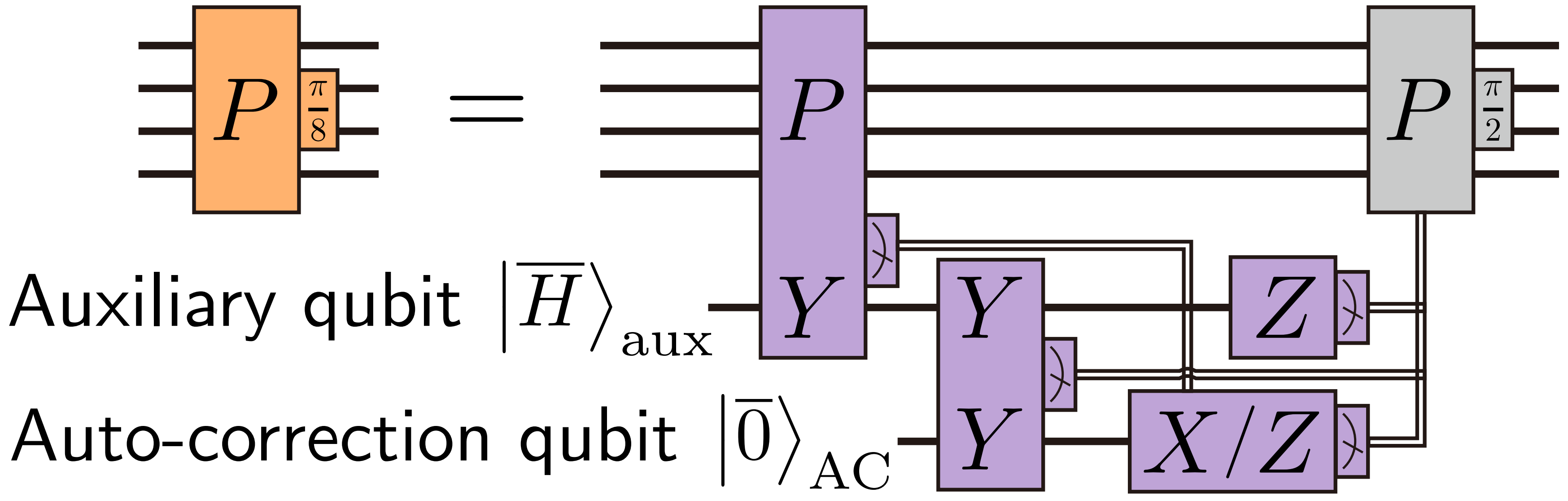
npj Quantum Information (2020)6:91 ; https://doi.org/10.1038/s41534-020-00319-5

Production of Higher-Quality Magic States

Combined MSD Scheme

- The CN protocol is executed repeatedly in each auxiliary patch.
- $|\overline{H}\rangle$ is first prepared in a patch with distance $d_{\mathrm{CN}} \leq 7$, then grown to distance d_{m}.
- Use more than two auxiliary patches.
  ∵ The CN protocol takes some time.



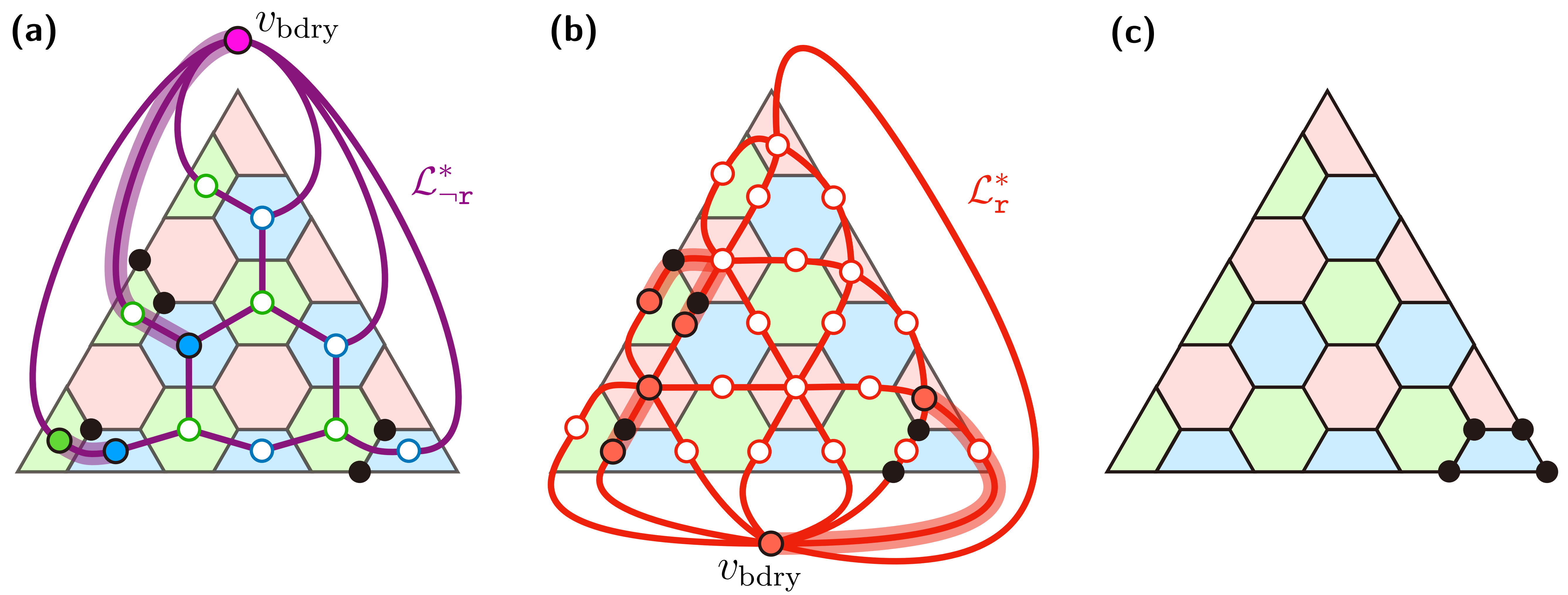


Performance analysis

- Suppose the circuit-level noise model with strength p
- Use the Concatenated MWPM decoder

Ref) Lee et al., *Color code decoder with improved scaling for correcting circuit-level noise*, arXiv:2404.07482 (2024).

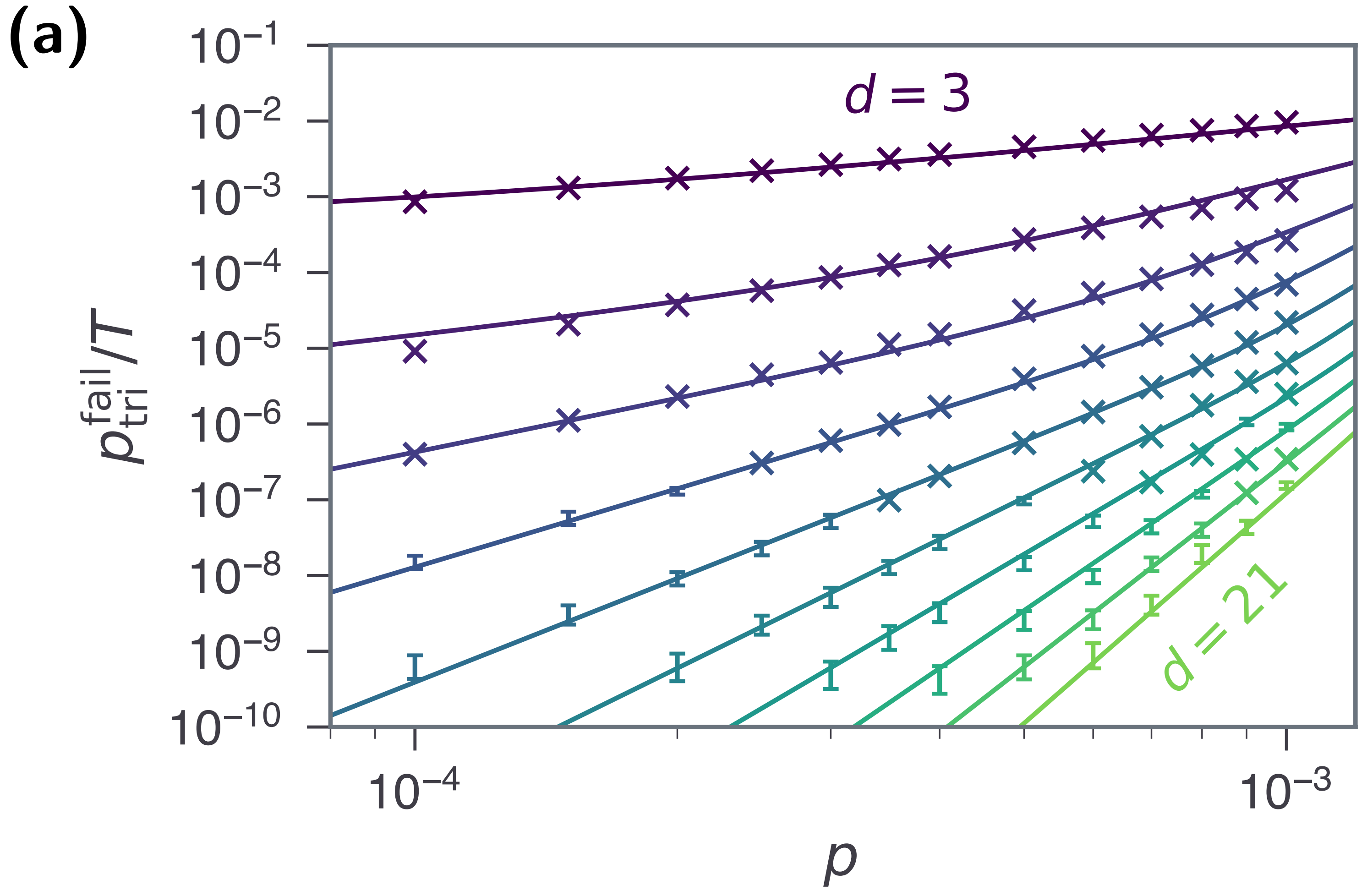
  - Circuit-level threshold of 0.46%
  - Nearly reaches the best sub-threshold scaling: $p_{\text{fail}} \sim p^{d/2}$

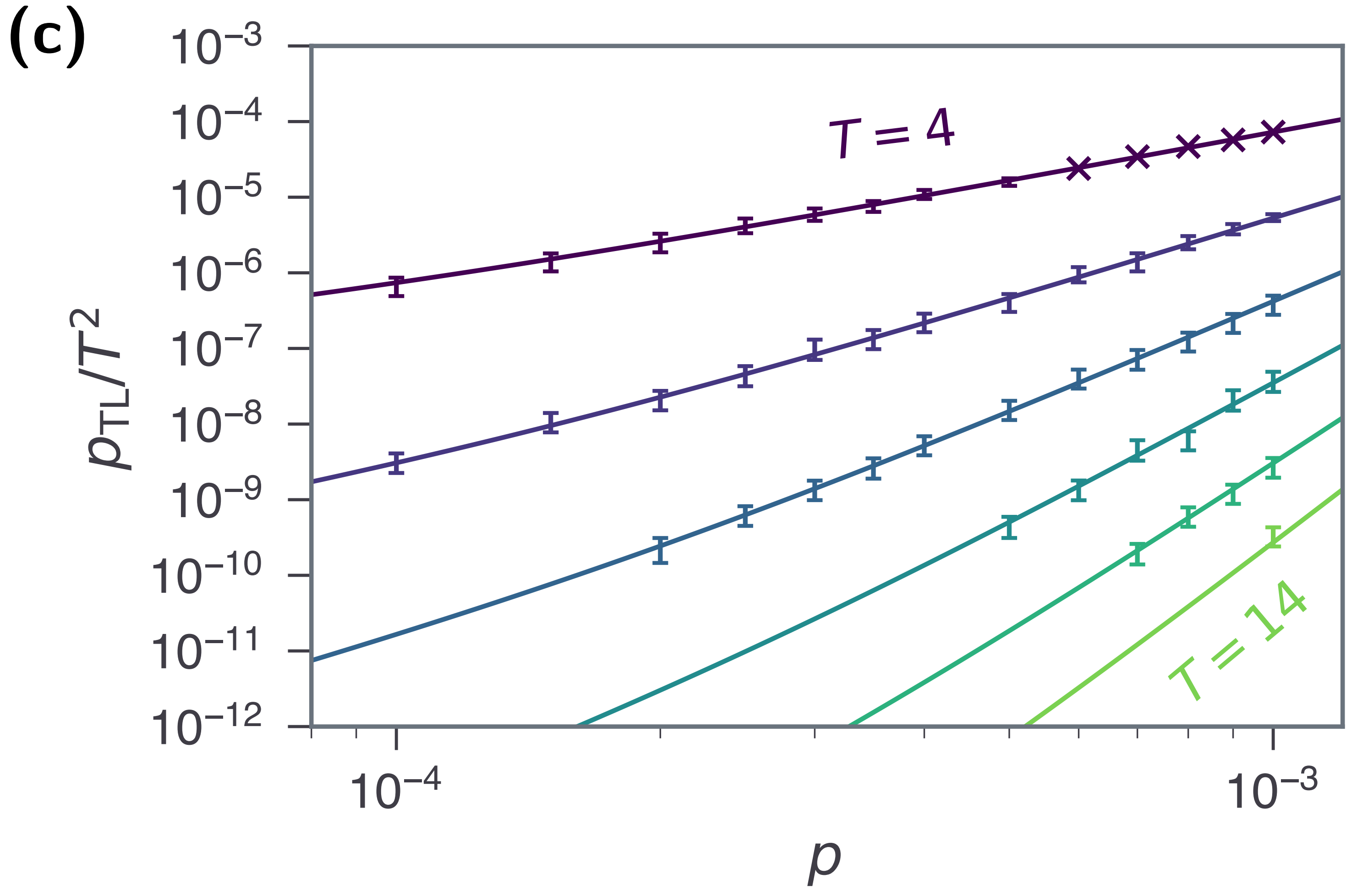


Performance analysis

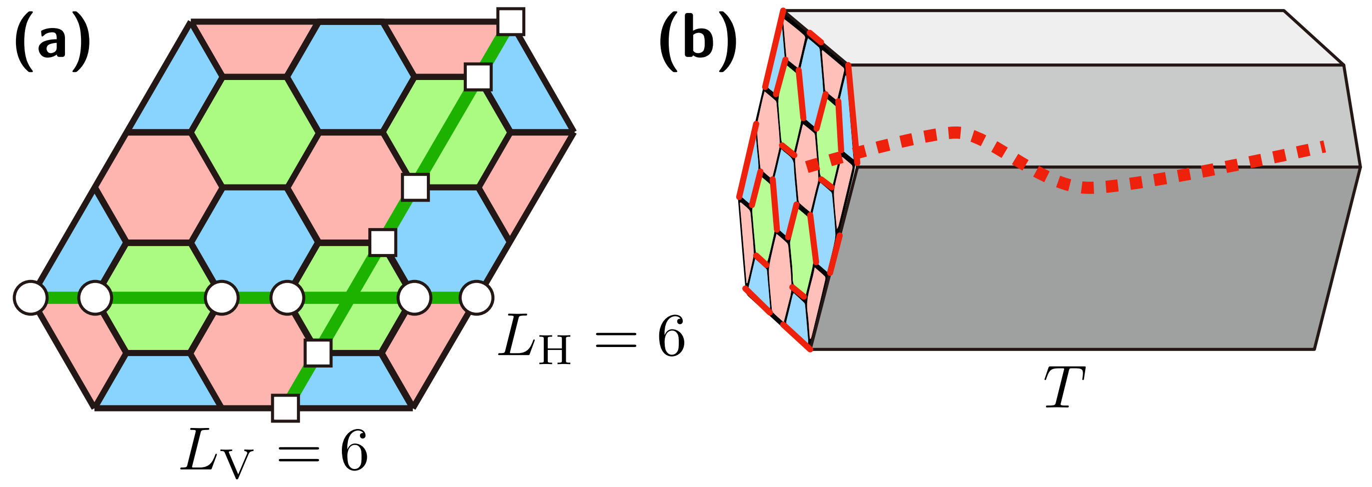
Logical error rates (per round/area) vs circuit-level noise

Triangular

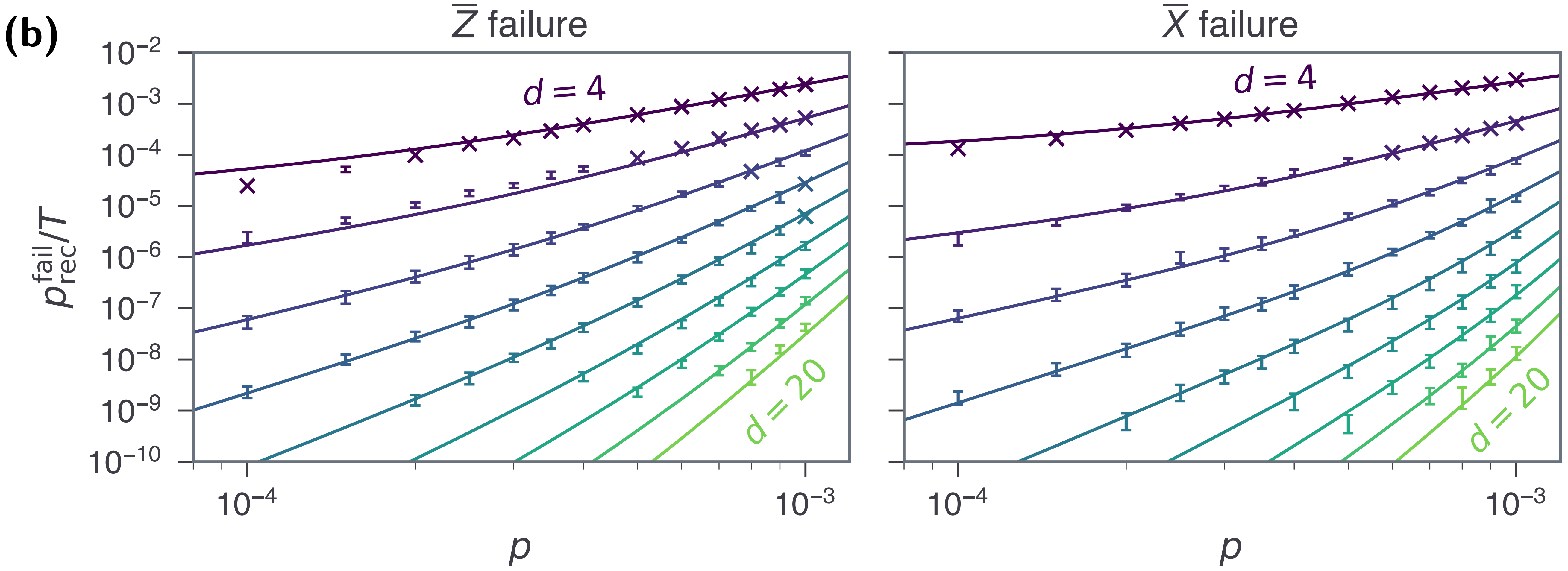




Stability experiment



Rectangular

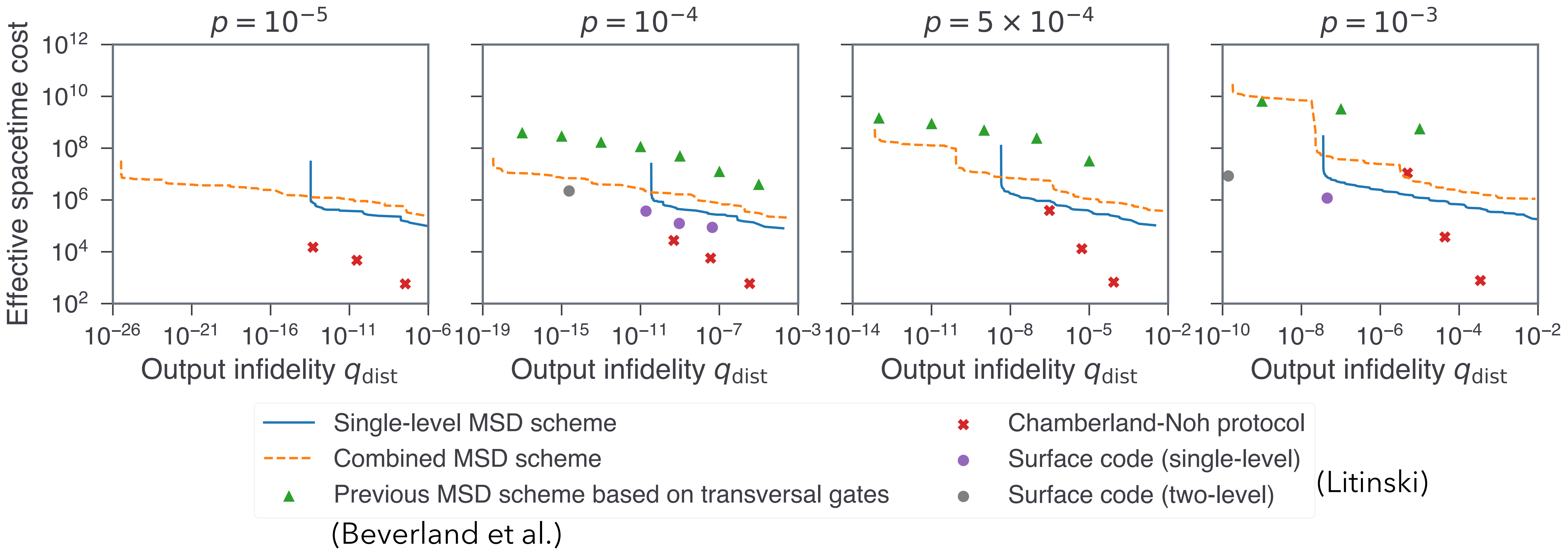


$$p_{\mathrm{L}}(p, d) = \alpha \left(\frac{p}{p_{\mathrm{th}}} \right)^{\beta d + \eta} \left[1 + \epsilon \left(\frac{p}{p_{\mathrm{th}}} \right)^{\zeta d^{\lambda}} \right]$$

Performance analysis

- Possible error sources
  - Triangular and rectangular patches
  - Ancillary region (for lattice surgery)
  - Non-Clifford components (faulty T-measurement or the Chamerland-Noh protocol)
- Each type of error source can be mapped to a noise channel of the form
  $\Lambda_{\overline{U},p_{\text{err}}}: \ \overline{\rho} \mapsto (1-p_{\text{err}})\overline{\rho} + p_{\text{err}}\overline{U}\overline{\rho}\overline{U}$
  acted on the ideal final state of the output and validation qubits
  - $\overline{\rho}$: logical state of the output and validation qubits
  - p_{err}: logical error rate
    (obtained from simulating triangular/rectangular patches with the concatenated MWPM decoder)
  - $\overline{U}$: product of $\pi/2$- or $(\pm\pi/4)$-rotations in bases consisting of $\overline{Z}$ operators only

Performance analysis



- The single-level MSD scheme reaches $q_{\mathrm{dist}} \gtrapprox 35p^3$ as expected.
- The combined MSD scheme reaches very low output infidelities, e.g., $\sim 3.4 \times 10^{-19}$ when $p = 10^{-4}$.
- Compared to a previous color code MSD scheme by Beverland et al. Ref) PRXQuantum.2.020341 (2021)
  : ~ 40 times improvement in spacetime cost (#qubits $\times$ #timesteps) when $p = 10^{-4}$
- Still not better than the surface code scheme

Performance analysis

Scheme	Output infidelity q_{dist}	Failure rate $1-q_{\text{succ}}$	Space cost n (Qubits)	Time cost t (Time steps)	Effective spacetime cost (nt/q_{succ})
			(a) $p=10^{-4}$		
sng-$(9,4,6,3)$	1.56×10^{-6}	3.83×10^{-2}	569	256	1.51×10^{5}
sng-$(9,6,6,5)$	1.94×10^{-7}	2.84×10^{-3}	701	384	2.70×10^{5}
sng-$(11,8,6,5)$	2.62×10^{-8}	2.91×10^{-3}	833	384	3.21×10^{5}
sng-$(13,6,8,5)$	1.83×10^{-9}	2.31×10^{-3}	1093	384	4.21×10^{5}
sng-$(17,10,12,5)$	1.40×10^{-10}	2.37×10^{-3}	2193	384	8.44×10^{5}
sng-$(21,14,12,9)$	3.54×10^{-11}	1.50×10^{-3}	2917	640	1.87×10^{6}
cmb-$(17,16,12,5,5,3)$	1.46×10^{-11}	1.11×10^{-3}	3441	750	2.59×10^{6}
cmb-$(17,12,12,7,5,5)$	2.98×10^{-13}	1.47×10^{-4}	5717	684	3.92×10^{6}
cmb-$(21,14,14,7,7,6)$	1.05×10^{-15}	3.62×10^{-5}	7583	940	7.13×10^{6}
cmb-$(23,16,16,9,7,3)$	1.82×10^{-17}	4.55×10^{-6}	6979	1539	1.07×10^{7}
cmb-$(29,22,20,13,7,4)$	3.40×10^{-19}	3.20×10^{-6}	1.47×10^{4}	1597	2.36×10^{7}
			(b) $p=5\times10^{-4}$		
sng-$(11,8,6,5)$	1.33×10^{-5}	3.88×10^{-2}	833	384	3.33×10^{5}
sng-$(15,6,10,5)$	1.31×10^{-6}	2.96×10^{-2}	1477	384	5.84×10^{5}
sng-$(17,8,10,7)$	1.05×10^{-7}	1.01×10^{-2}	1785	512	9.23×10^{5}
sng-$(21,10,12,9)$	1.15×10^{-8}	7.93×10^{-3}	2645	640	1.71×10^{6}
sng-$(29,18,18,15)$	4.43×10^{-9}	7.52×10^{-3}	6053	1024	6.25×10^{6}
cmb-$(23,16,14,9,5,6)$	1.02×10^{-9}	2.47×10^{-3}	1.01×10^{4}	1167	1.18×10^{7}
cmb-$(29,22,18,11,5,6)$	1.00×10^{-10}	2.08×10^{-3}	1.51×10^{4}	1228	1.86×10^{7}
cmb-$(31,18,20,11,7,5)$	1.19×10^{-11}	3.69×10^{-4}	1.39×10^{4}	9097	1.27×10^{8}
cmb-$(35,20,24,11,7,5)$	1.50×10^{-12}	3.72×10^{-4}	1.56×10^{4}	9097	1.42×10^{8}
cmb-$(41,26,28,13,7,6)$	1.09×10^{-13}	2.17×10^{-4}	2.40×10^{4}	8932	2.14×10^{8}
cmb-$(47,32,26,21,7,5)$	6.97×10^{-14}	1.89×10^{-4}	3.90×10^{4}	9209	3.59×10^{8}
			(c) $p=10^{-3}$		
sng-$(17,8,12,7)$	1.72×10^{-5}	5.37×10^{-2}	2149	512	1.16×10^{6}
sng-$(23,16,14,9)$	1.13×10^{-6}	2.52×10^{-2}	3701	640	2.43×10^{6}
sng-$(29,20,18,11)$	1.27×10^{-7}	1.82×10^{-2}	5909	768	4.62×10^{6}
sng-$(47,24,26,23)$	3.58×10^{-8}	1.51×10^{-2}	1.31×10^{4}	1536	2.04×10^{7}
cmb-$(47,26,30,17,7,5)$	1.15×10^{-8}	6.99×10^{-3}	3.06×10^{4}	2.28×10^{5}	7.03×10^{9}
cmb-$(53,38,34,19,7,5)$	1.04×10^{-9}	4.21×10^{-3}	3.99×10^{4}	2.29×10^{5}	9.18×10^{9}
cmb-$(67,38,34,33,7,5)$	1.82×10^{-10}	2.59×10^{-3}	7.06×10^{4}	2.26×10^{5}	1.60×10^{10}

Take-Home Messages

- We need **end-to-end MSD schemes optimised for individual QEC codes**, not just their high-level structures.
- For color codes, we can exploit their advantages:
  - Simultaneous measurement of commuting Pauli operators via lattice surgery
  - Various types of logical patches (triangular / rectangular)
  - Transversal Clifford gate (which enables the Chamberland-Noh protocol)
- For realistic error analysis, **every possible type of logical error** that can happen during MSD should be considered:
  - Patches, ancillary regions, domain walls, non-Clifford components, and so on.
  - Theoretical estimations (e.g., $p_{\mathrm{L}} \approx 35p^3$) are valid only when other logical errors are negligible. They are indeed **NOT** in most cases.
- Our schemes **improve resource cost by up to two orders of magnitude**, approaching the performance levels of surface codes.

Thank you

arXiv:2409.07707