Color code decoder with improved scaling for correcting circuit-level noise

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Based on arXiv:2404.07482

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THE UNIVERSITY OF SYDNEY

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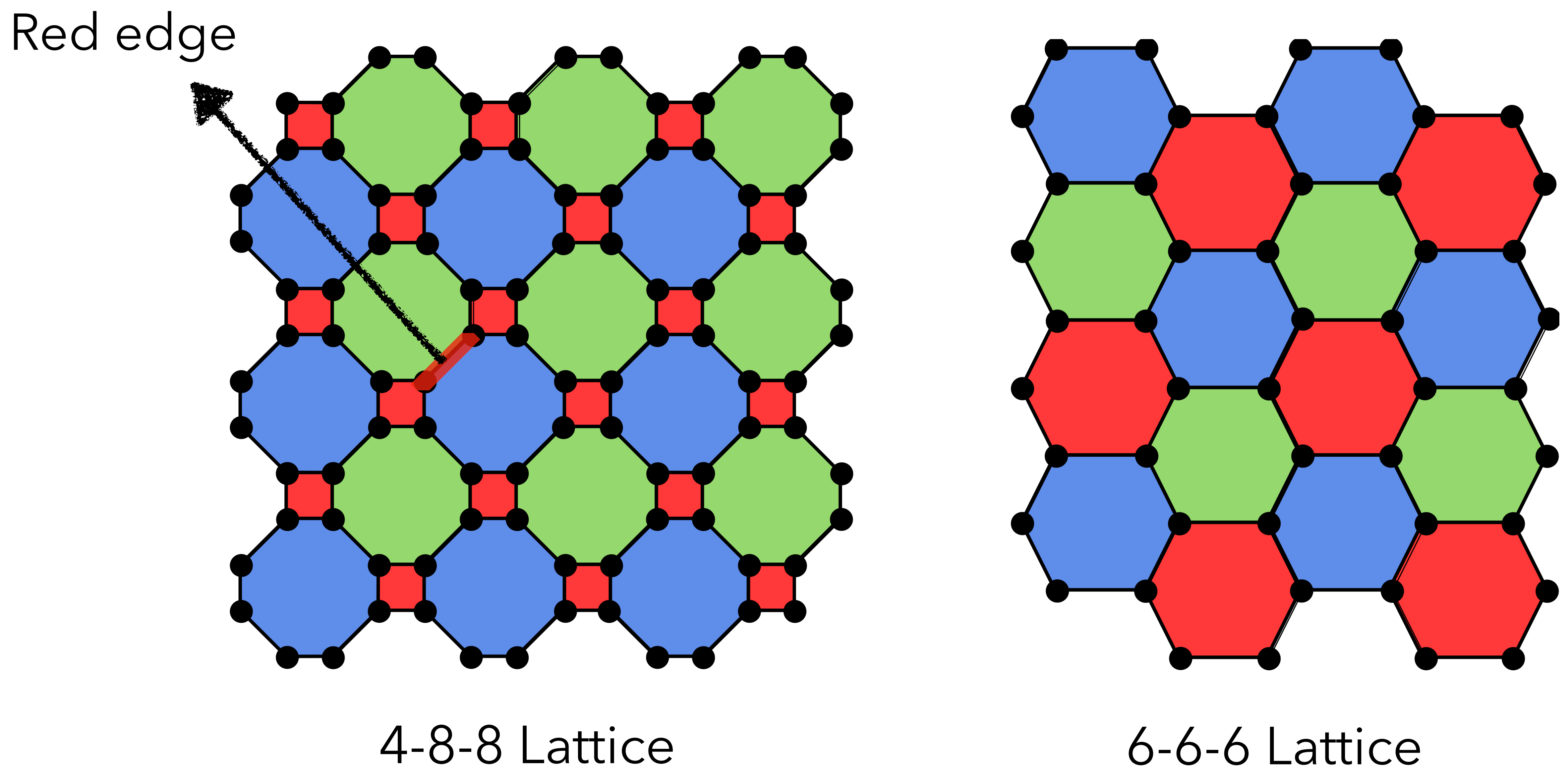
EQUS
Australian Research Council Centre of Excellence for Engineered Quantum Systems

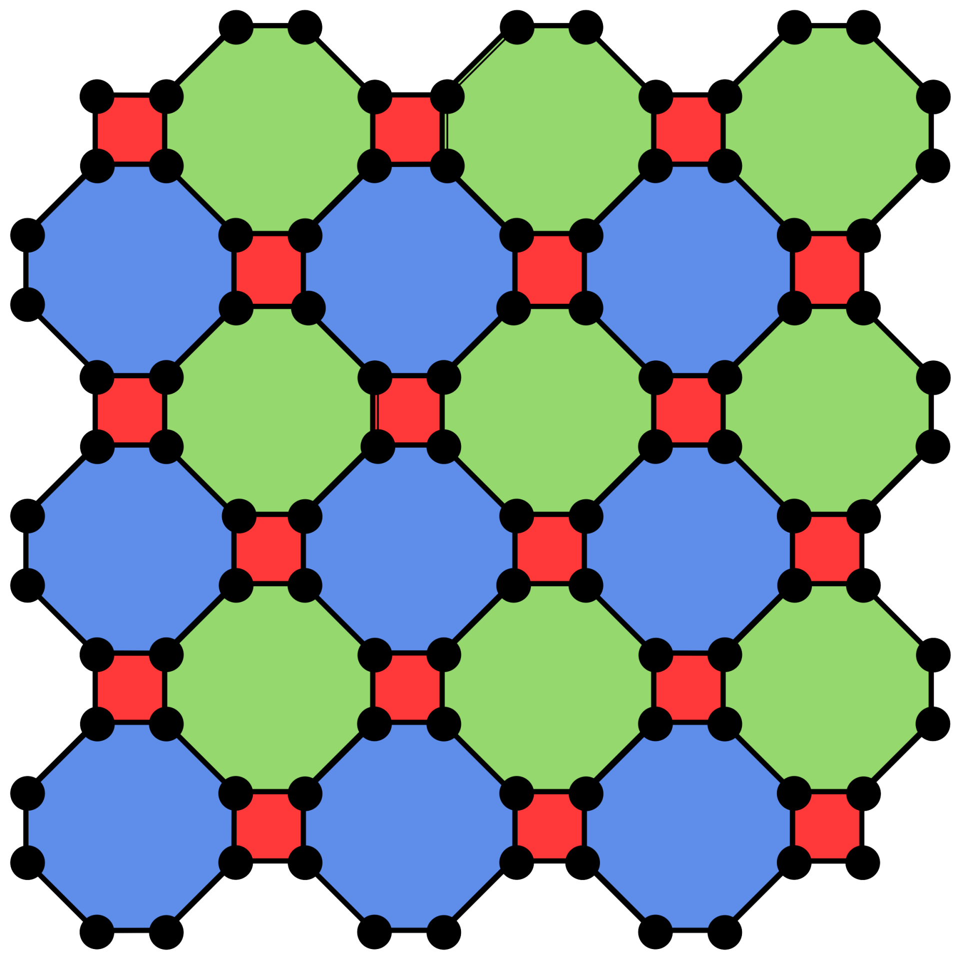
2D Color Codes

- Color-code Lattice [Bombin & Martin-Delgado, PRL 2006]
  - **3-valent**: Each vertex is connected with three edges.
  - **3-colorable**: Each face can be colored in one of three colors {r, g, b} in a way that adjacent faces do not have the same color.
- Edges are also colorable.

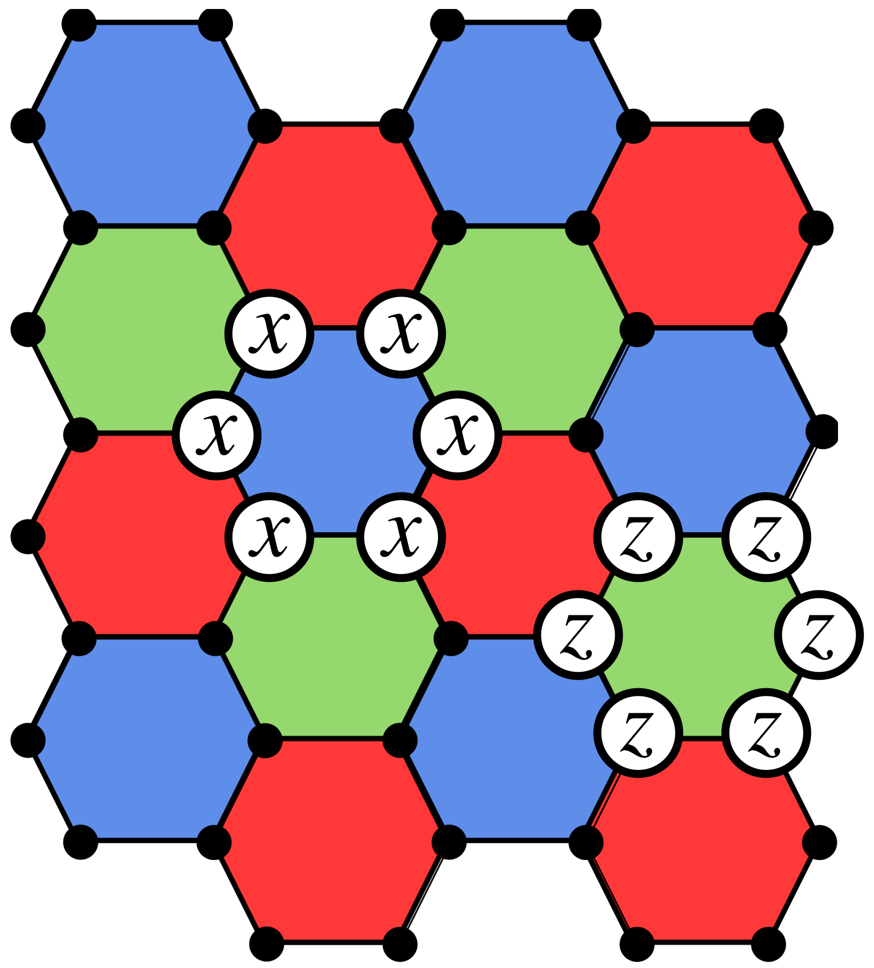


2D Color Codes

- Qubit on each vertex
- **Checks** (stabilizer generators):
  For each face f,
  - X-type check $S_f^X := \prod_{v \in f} X_v$
  - Z-type check $S_f^Z := \prod_{v \in f} Z_v$
  - $S_f^X |\psi\rangle = |\psi\rangle, \quad S_f^Z |\psi\rangle = |\psi\rangle$
- Contain only local connections



4-8-8 Lattice

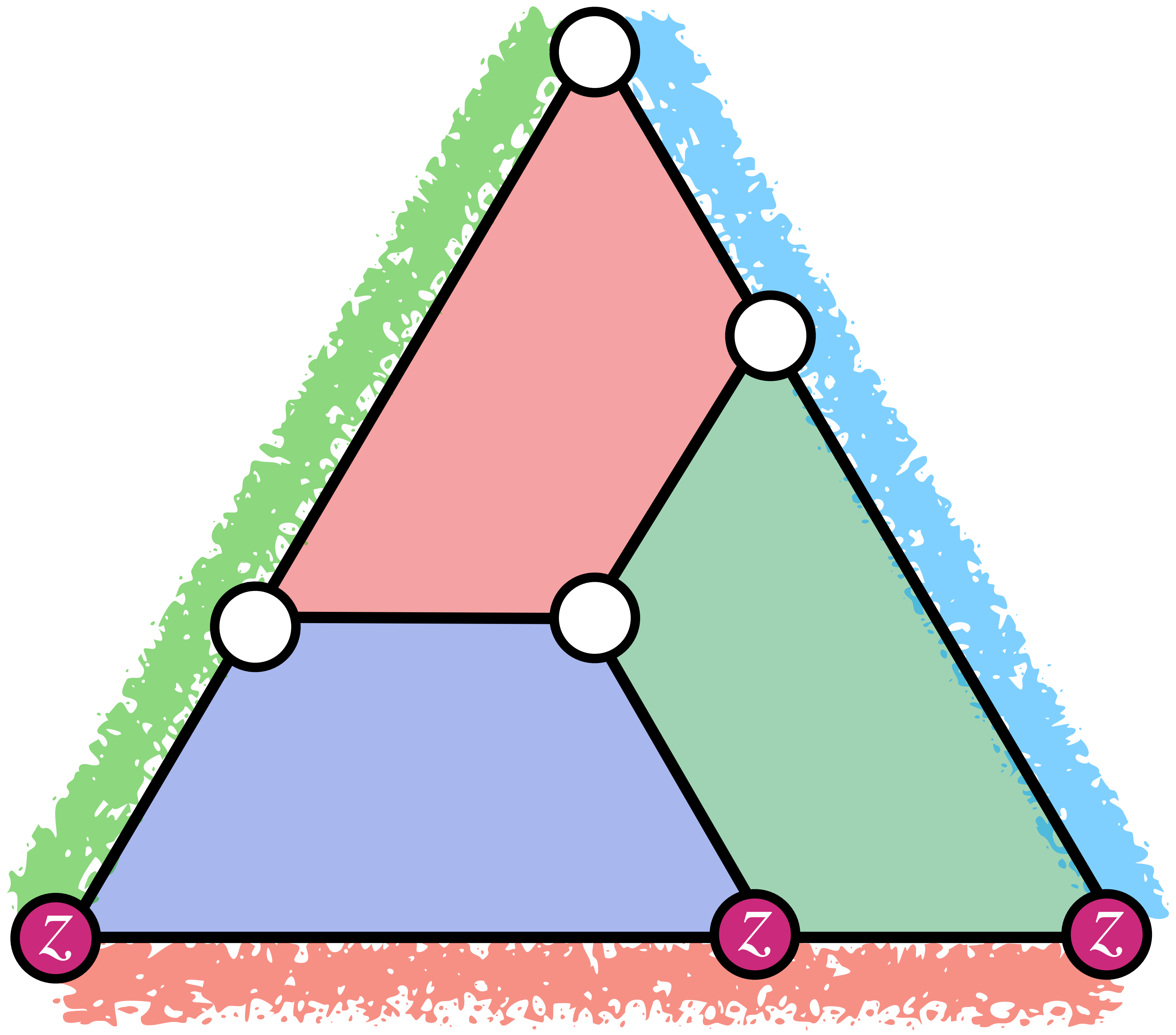


6-6-6 Lattice

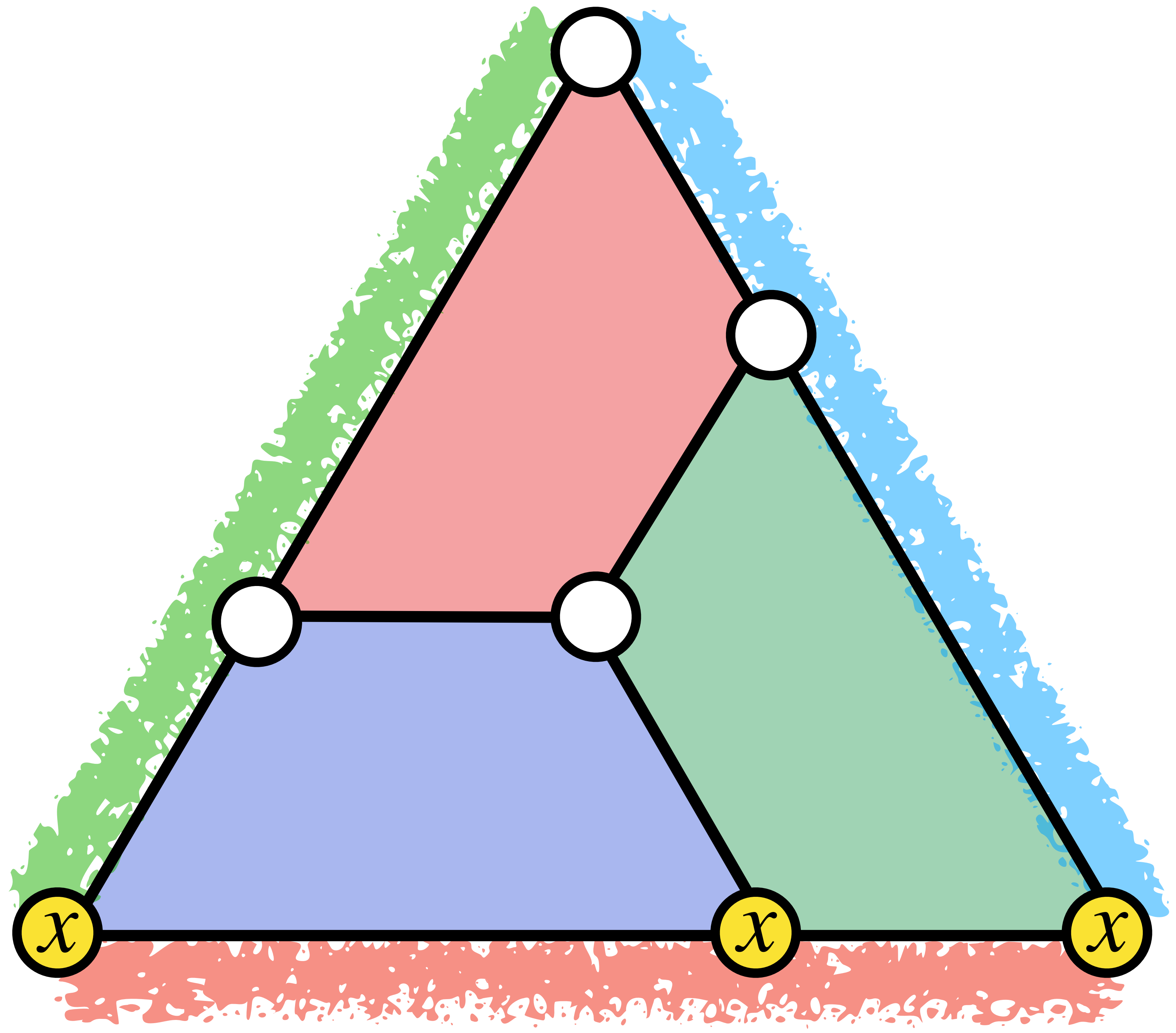
2D Color Codes

Logical qubits

- [[7, 1, 3]] triangular color code = 7-qubit Steane code



$\overline{Z}$ = Logical Z



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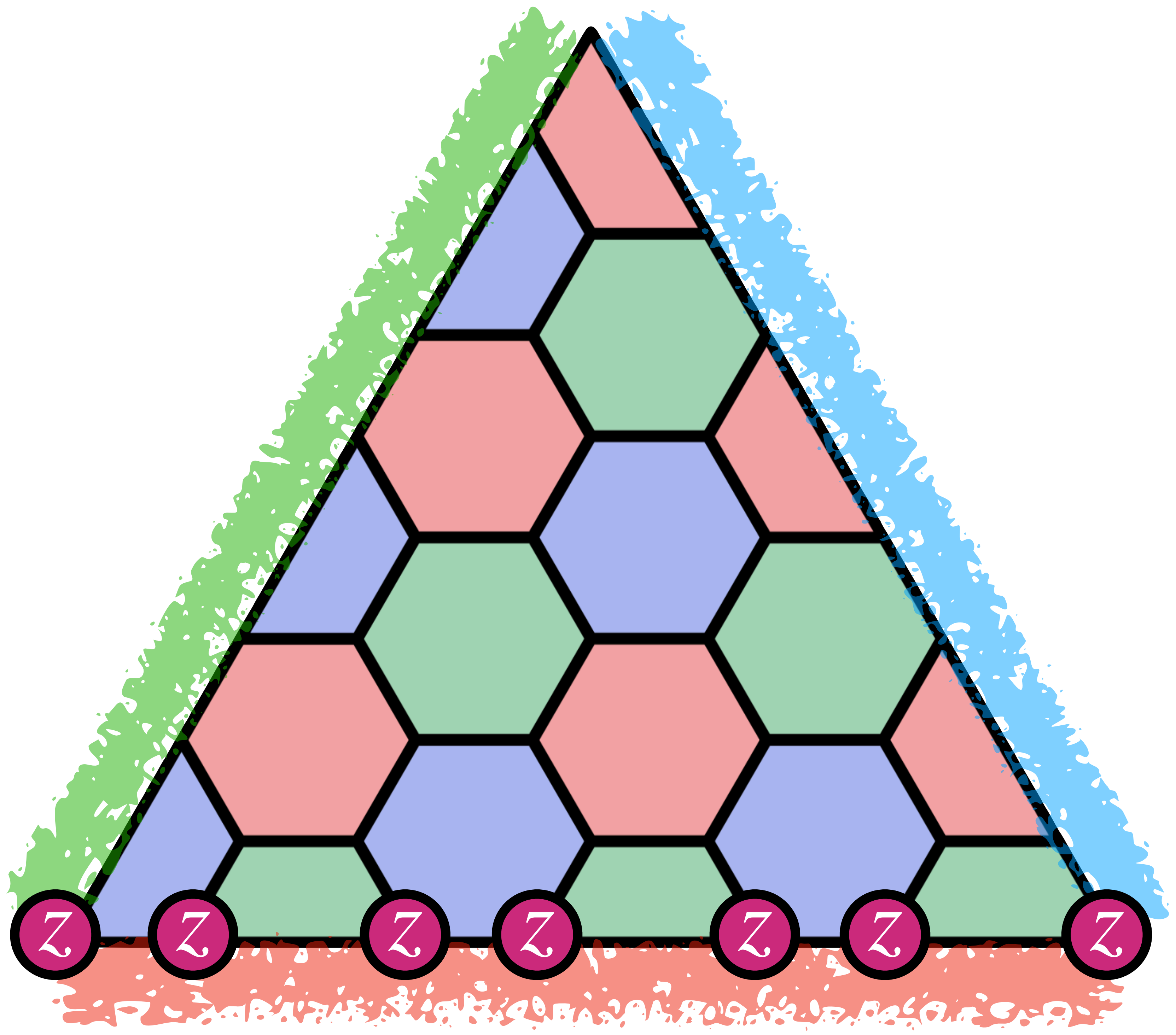
2D Color Codes

Logical qubits

- [[37, 1, 7]] triangular color code



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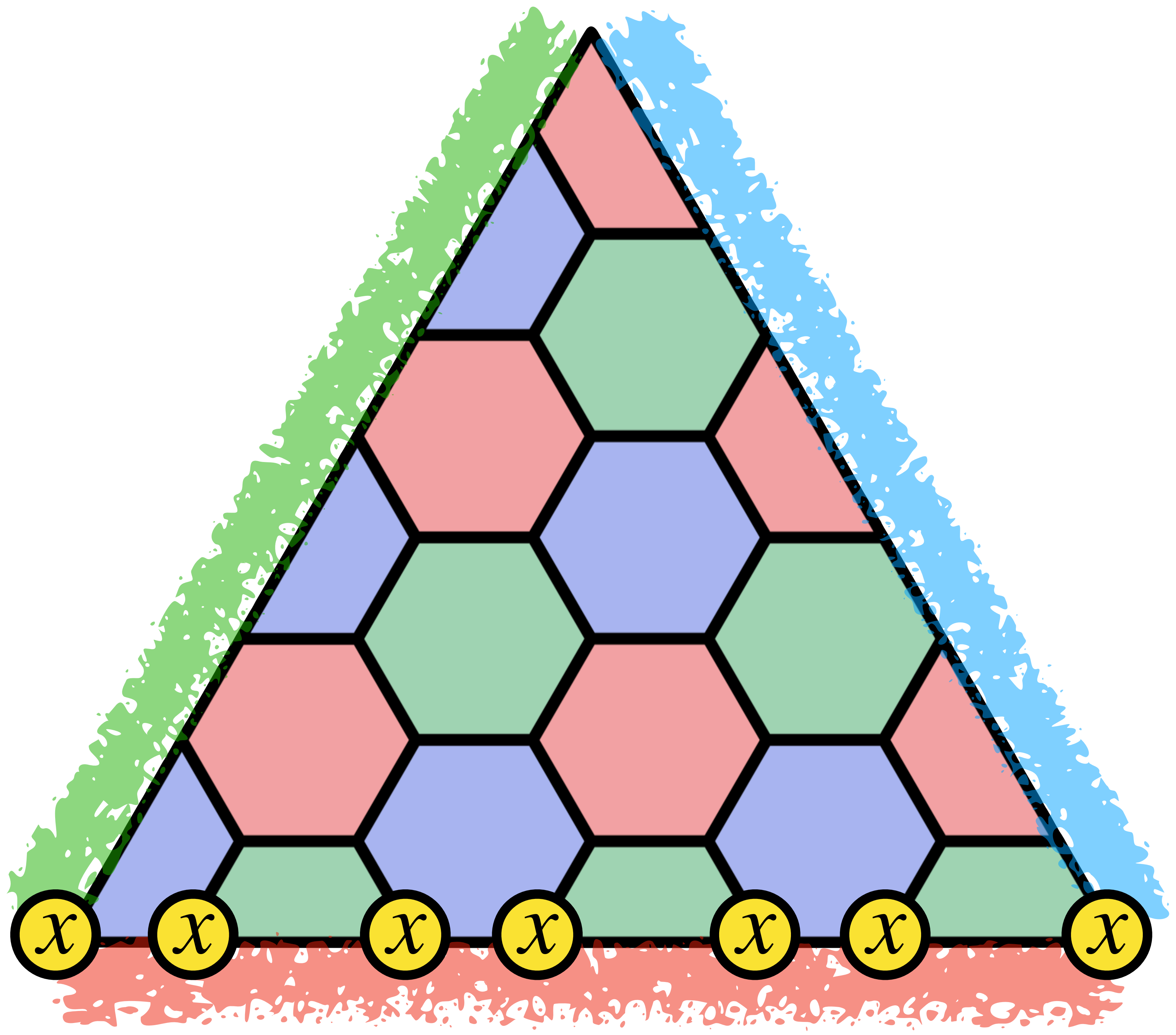


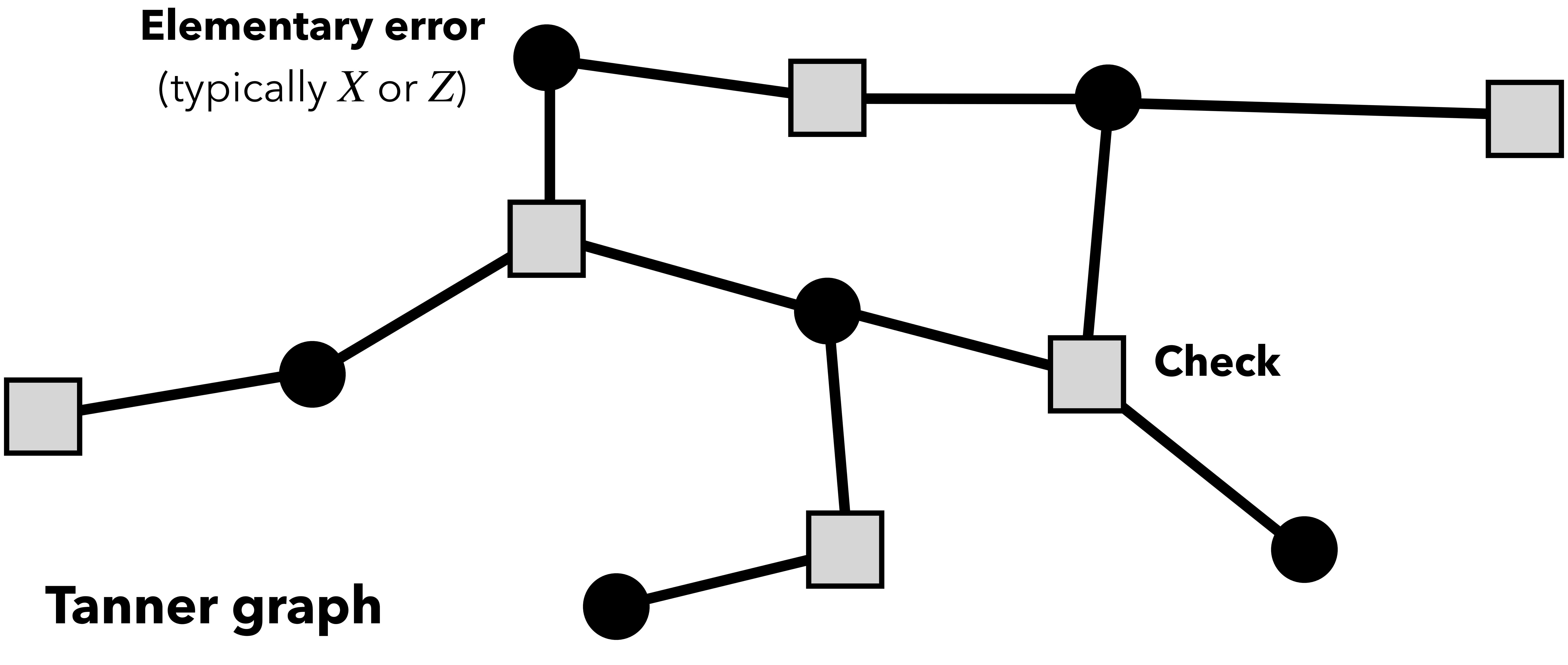
$\overline{X}$ = Logical X

2D Color Codes

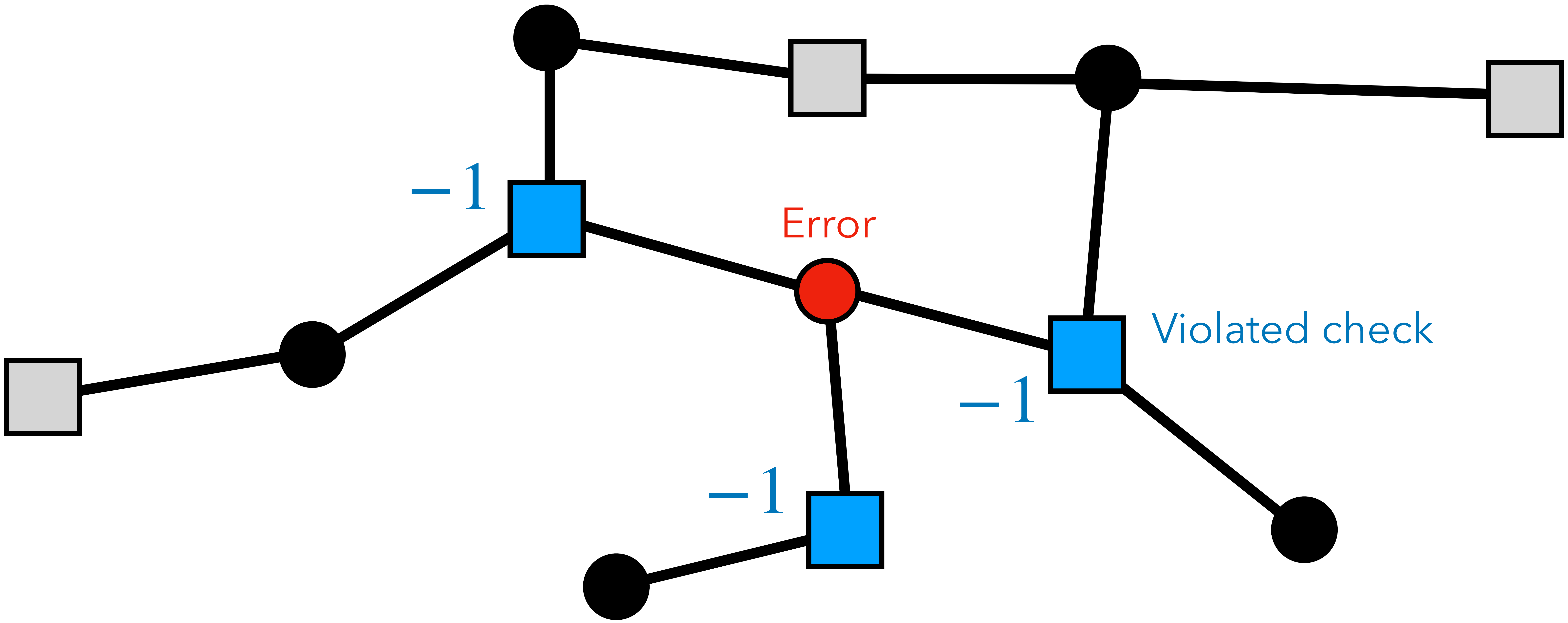
Why color codes?

- Advantages
  1. Lower spatial cost ($n/k \approx 3d^2/4$) than surface codes ($n/k = d^2$)
  2. Resource-efficient logical Pauli measurements using lattice surgery
     [Thomsen et al., arXiv:2201.07806]
  3. Transversal implementation of Clifford gates [Bombin & Martin-Delgado, PRL 2006]
- Disadvantages
  - Difficulty in decoding → Low fault-tolerance

Decoding Problem



Decoding Problem



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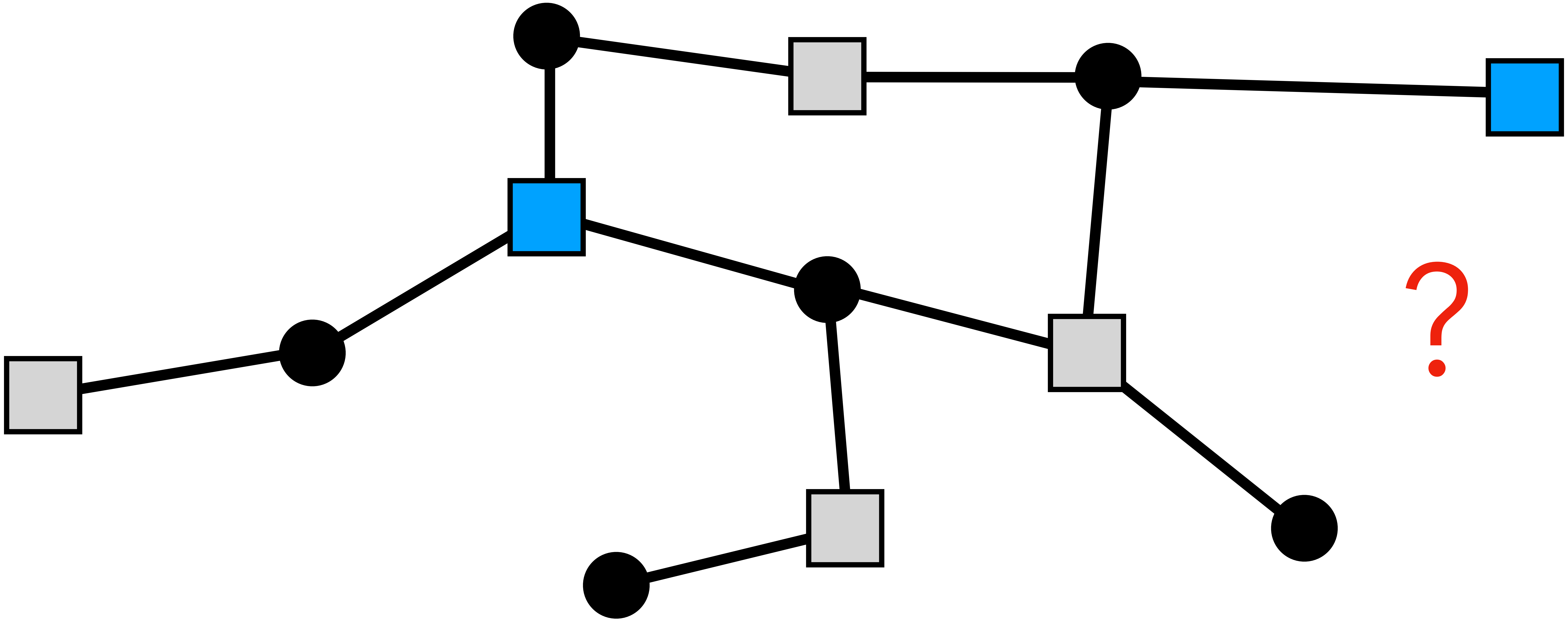
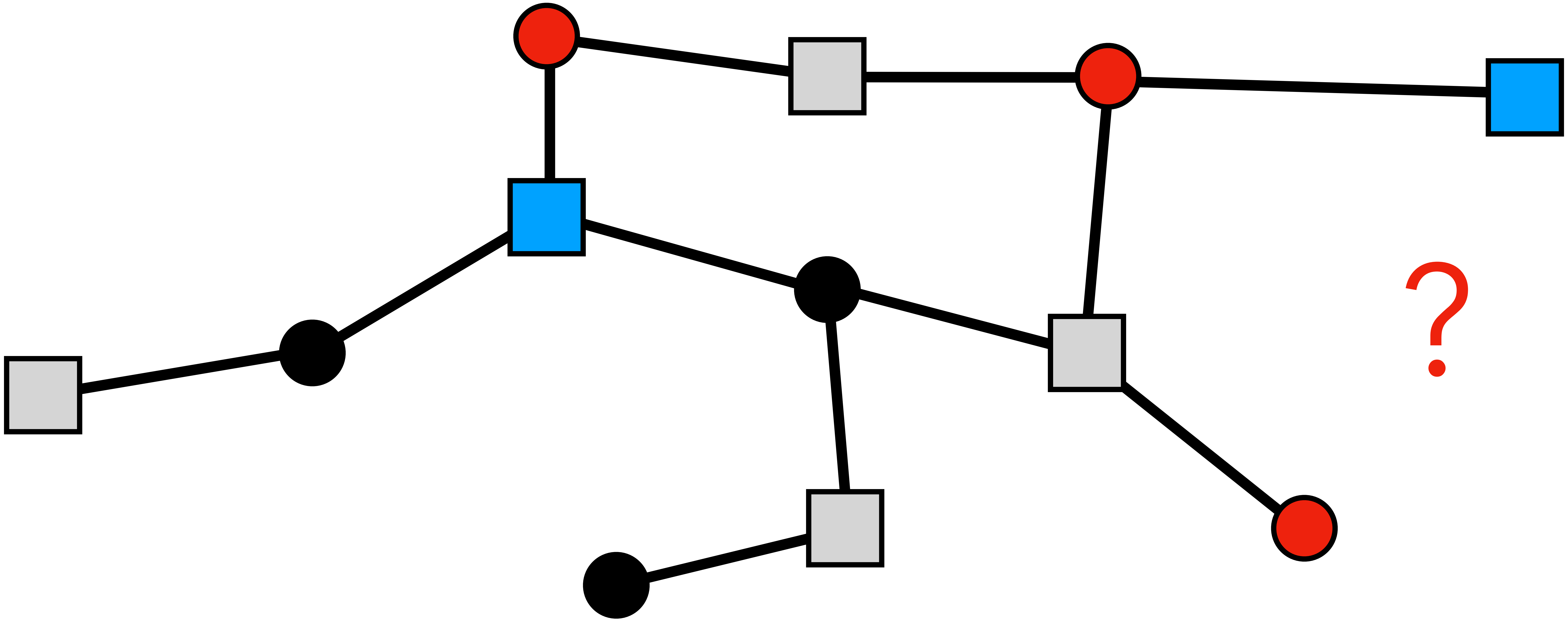


For given check outcomes, how to estimate errors?

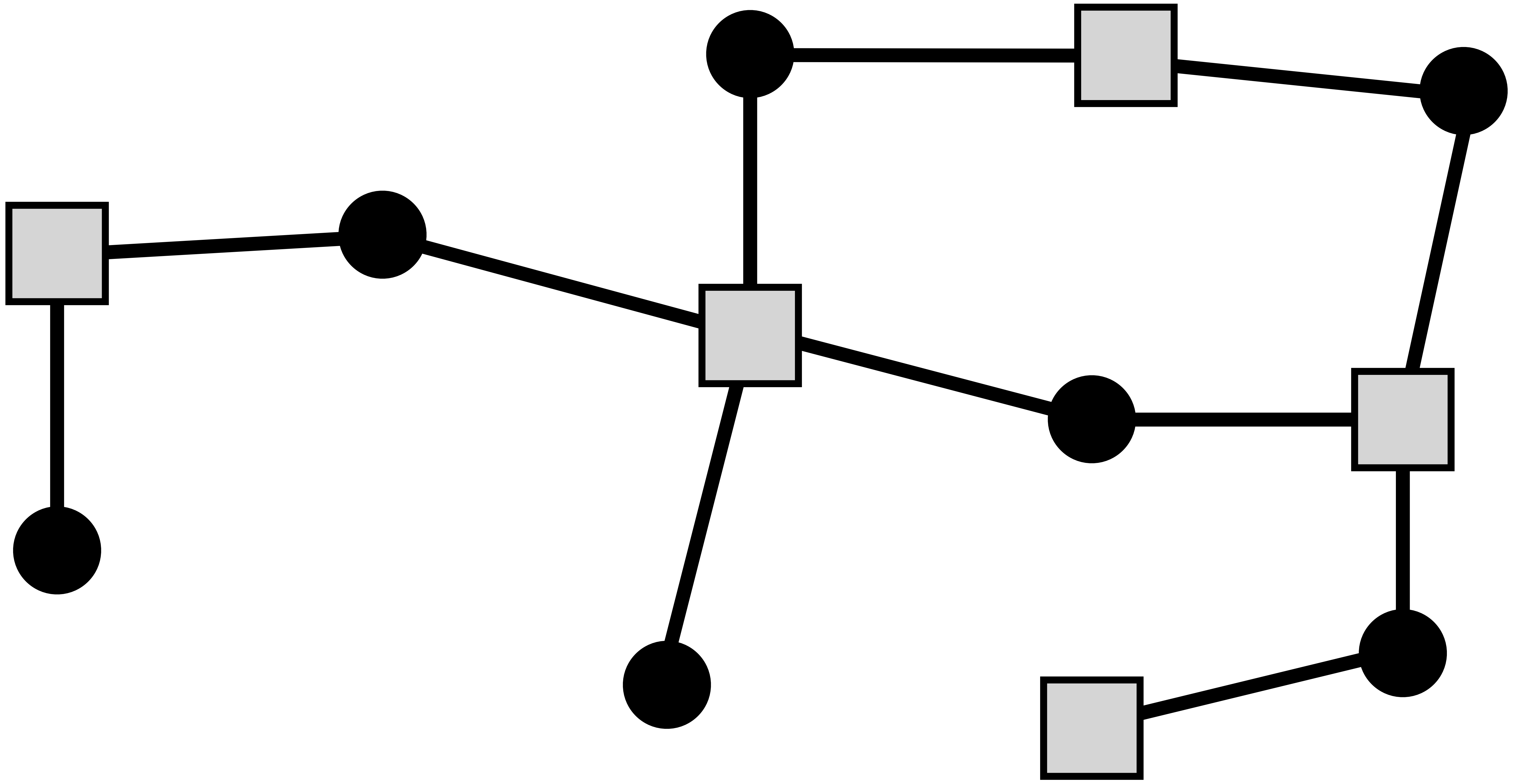
Decoding Problem



For given check outcomes, how to estimate errors?

Decoding Problem

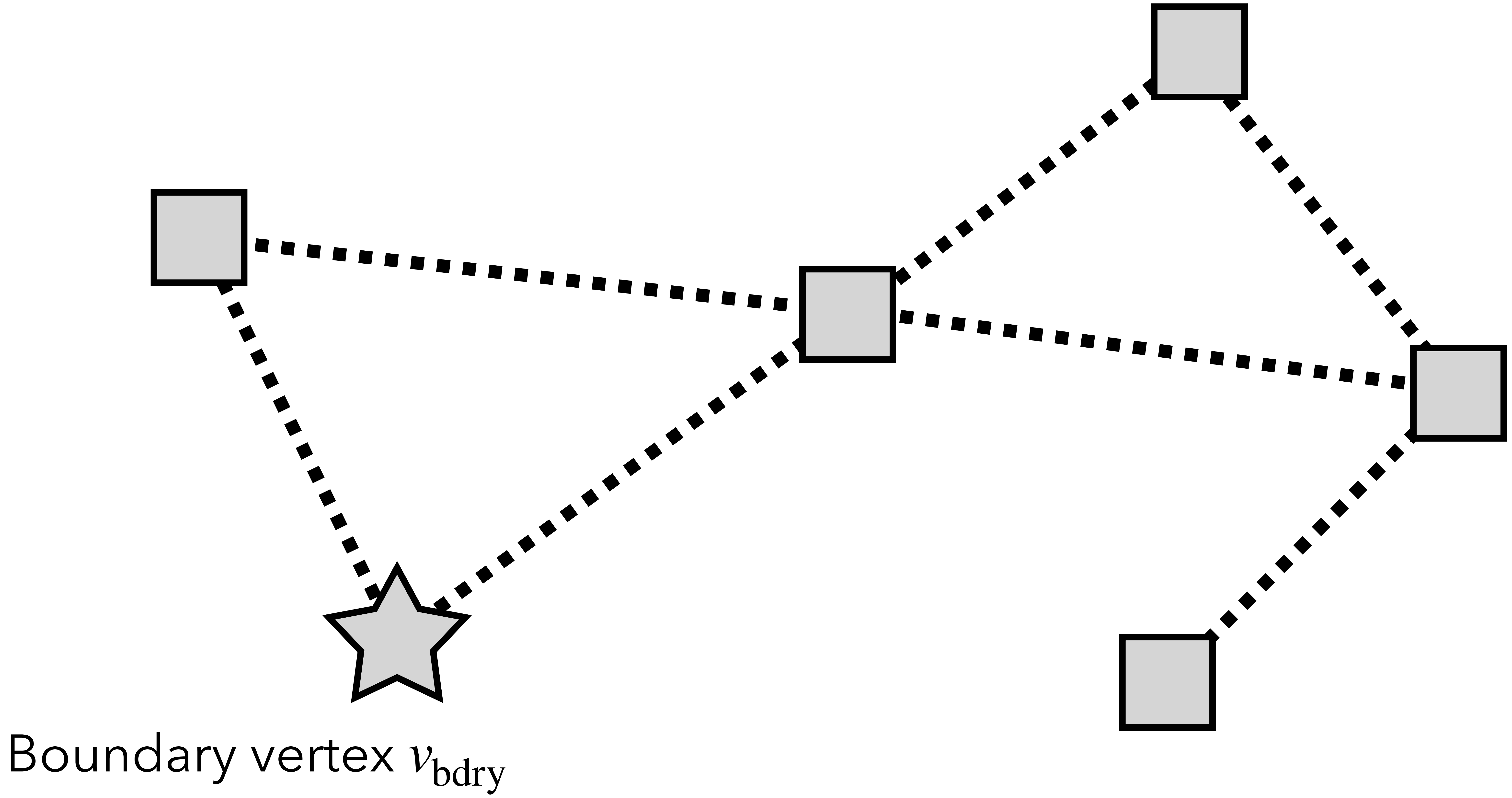
Minimum-weight perfect matching (MWPM)



Tanner graph containing only **edge-like elementary errors**

Decoding Problem

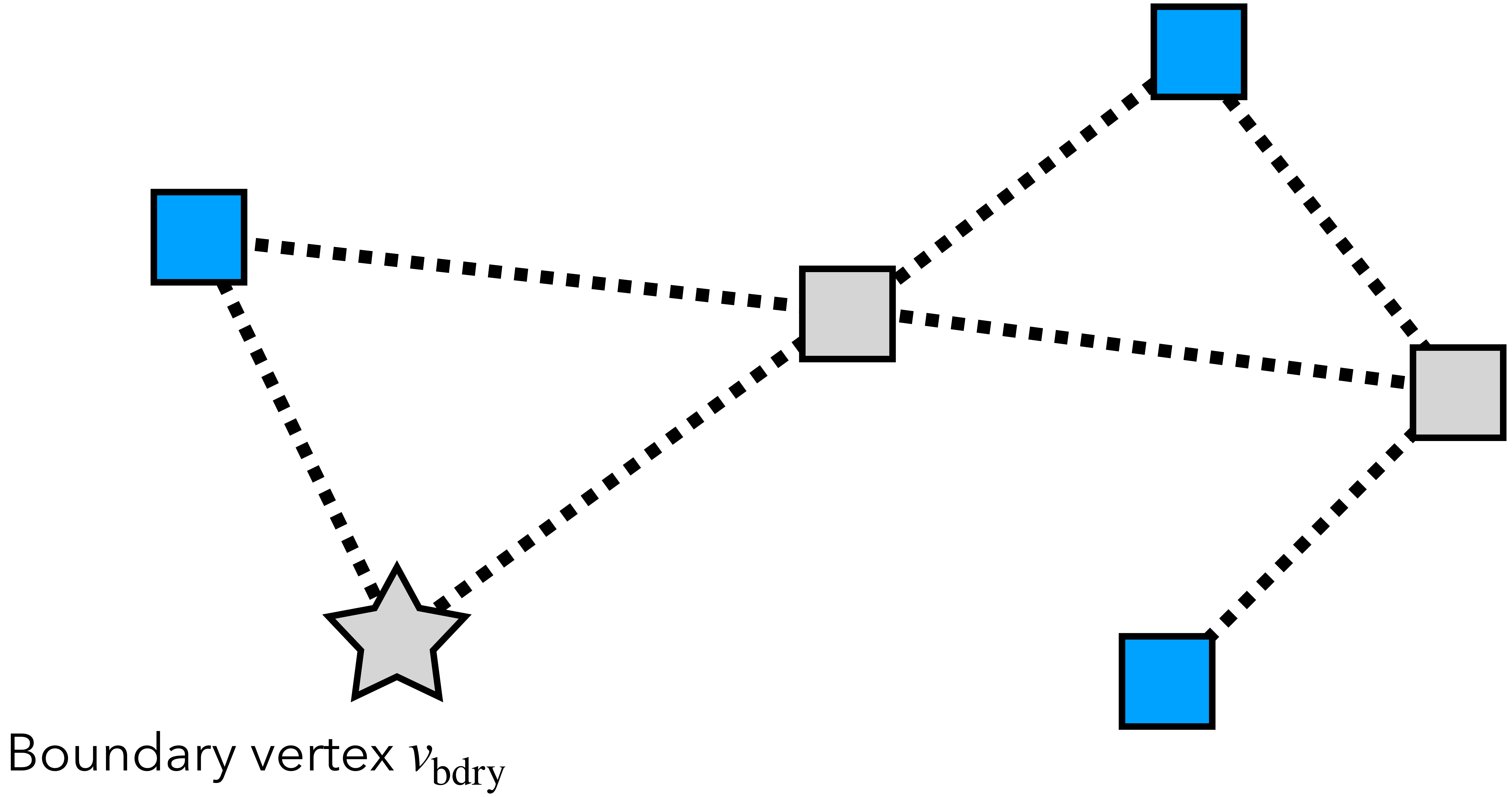
Minimum-weight perfect matching (MWPM)



Matching graph

Decoding Problem

Minimum-weight perfect matching (MWPM)



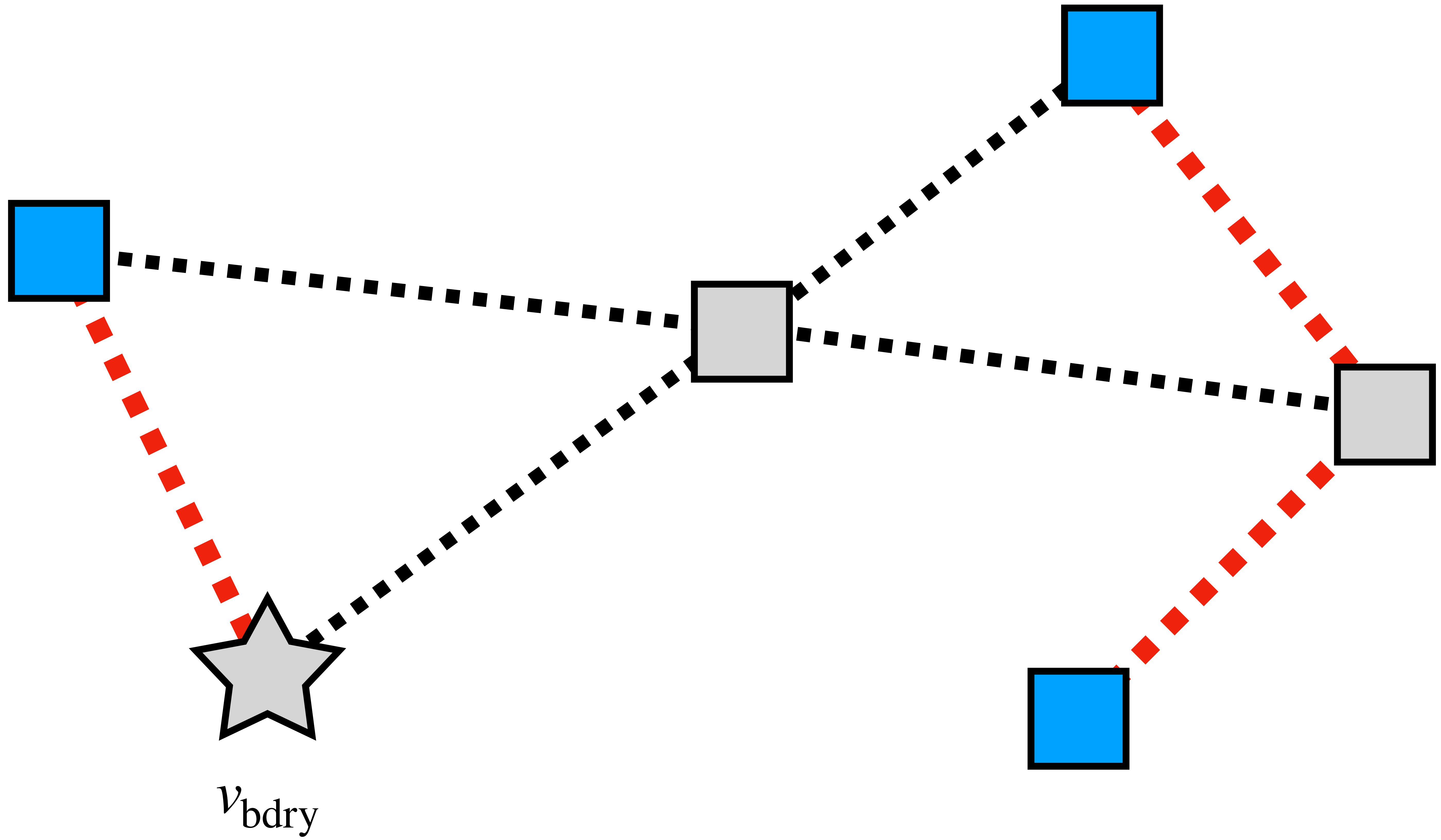
Matching graph

Decoding Problem

Minimum-weight perfect matching (MWPM)

- **Matching**
  : Set of edges that meet each violated/unviolated check an odd/even number of times (not considering v_{bdry})
- **MWPM decoder**
  : Find a matching that minimizes the sum of the *weights* of edges in it

$$w_e = \log\left[(1 - p_e)/p_e\right]$$



Decoding Problem

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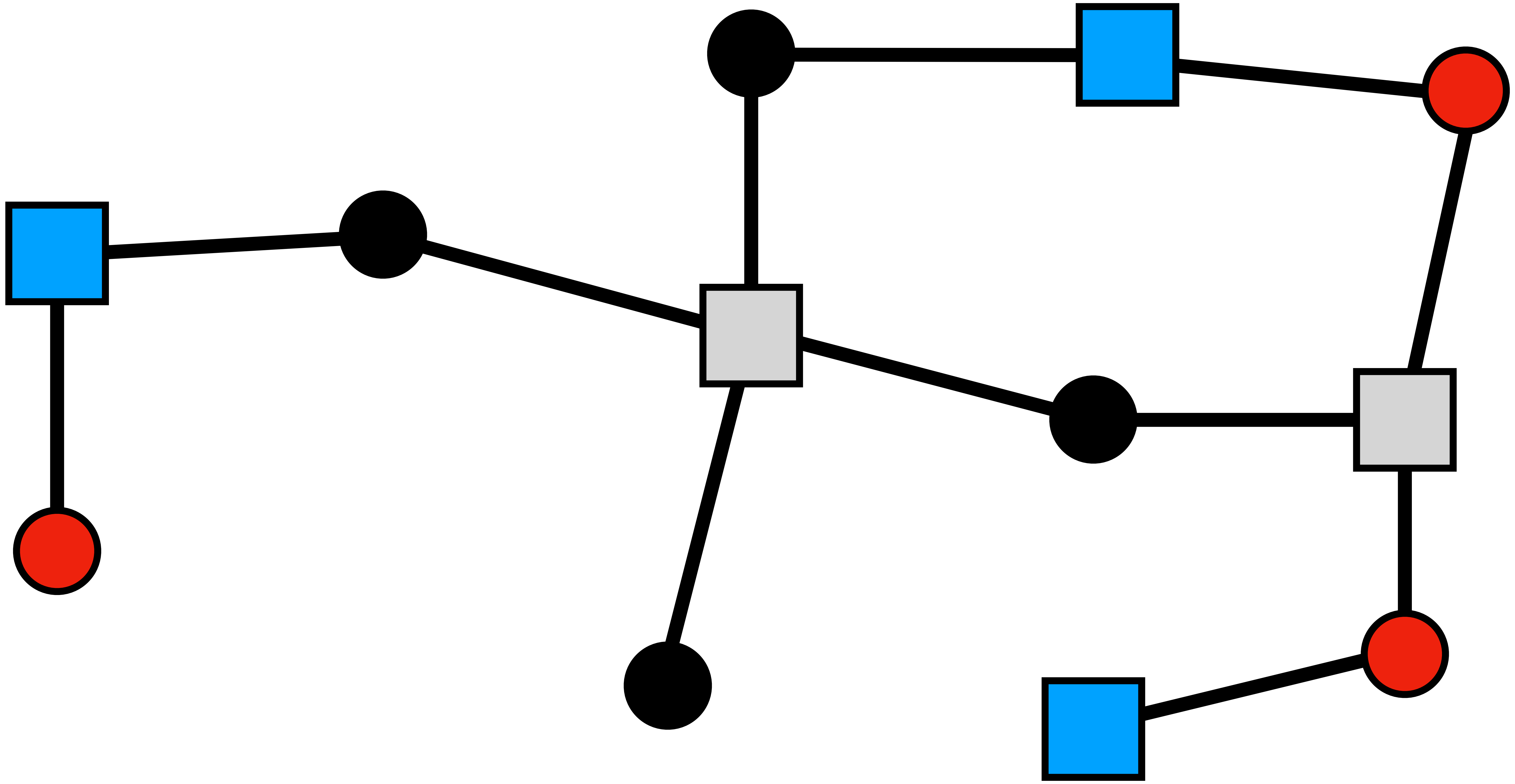

Decoding Problem

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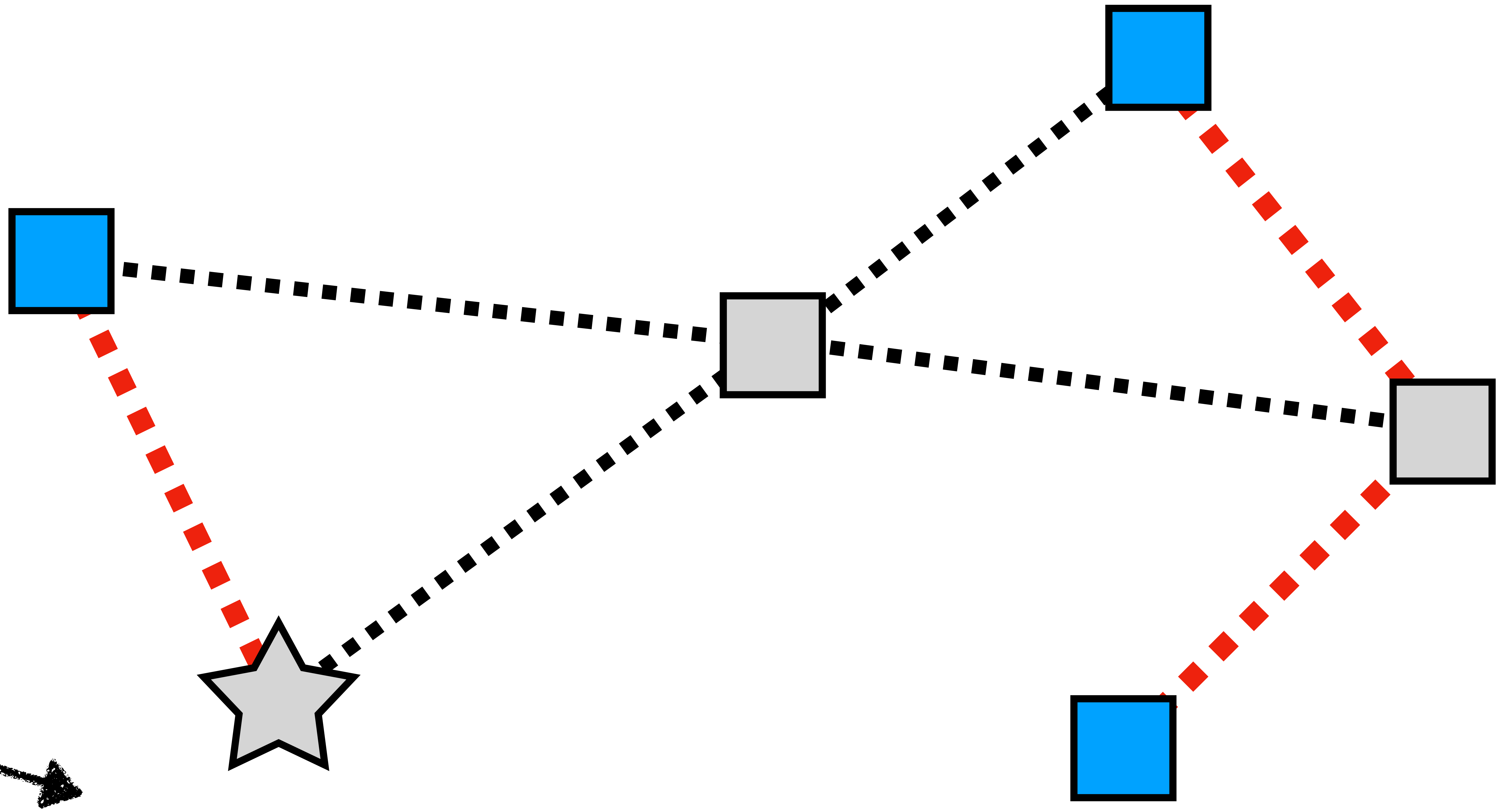
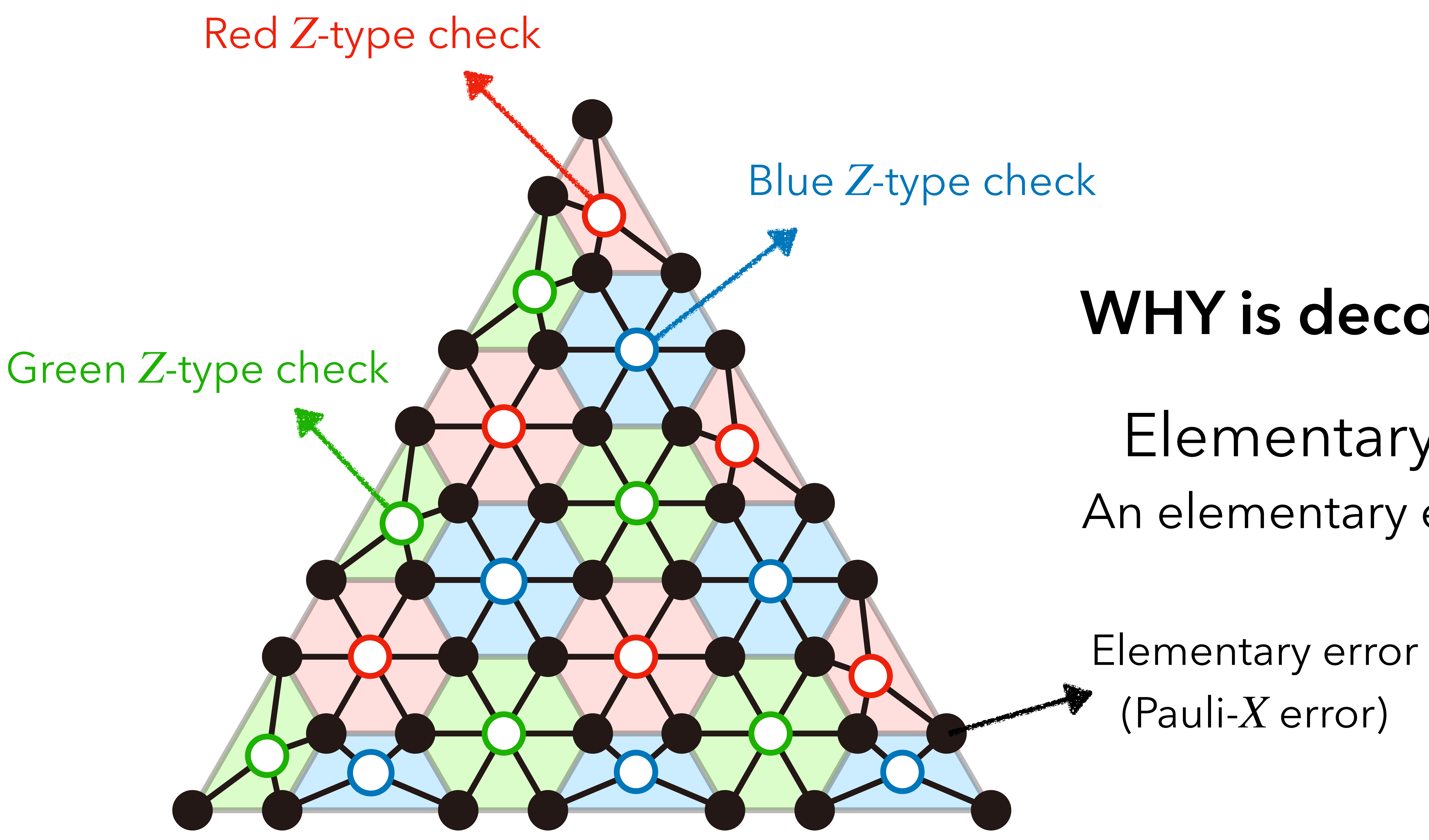


Most probable combination of elementary errors

$$\because \Pr(E) = \prod_{e\in E} p_e \prod_{e\notin E}(1-p_e) \propto \prod_{e\in E}\frac{p_e}{1-p_e} = \frac{1}{\exp\left(\sum_{e\in E} w_e\right)}$$

Decoding Color Codes



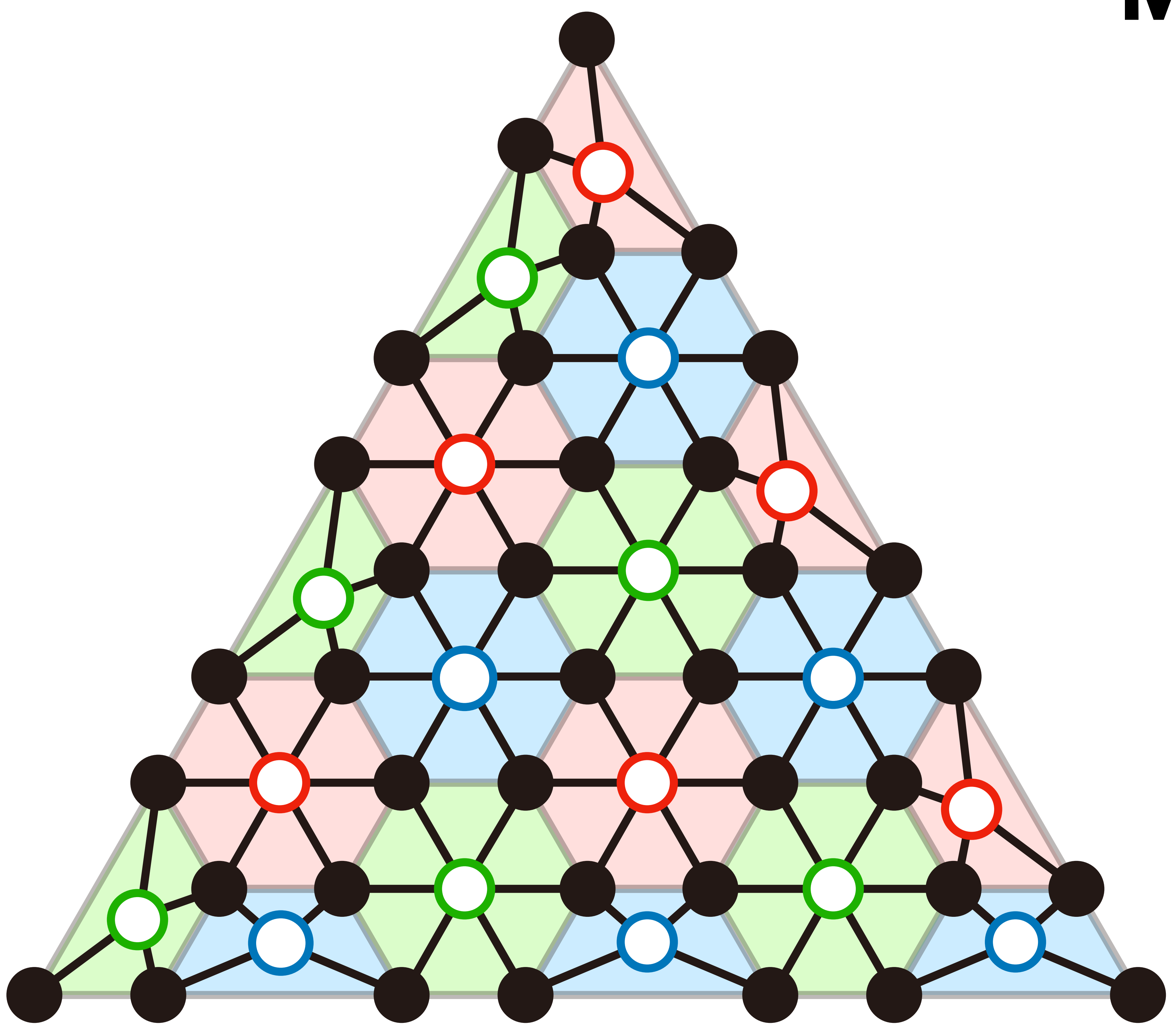
Tanner graph for Pauli-X errors

WHY is decoding color codes difficult?

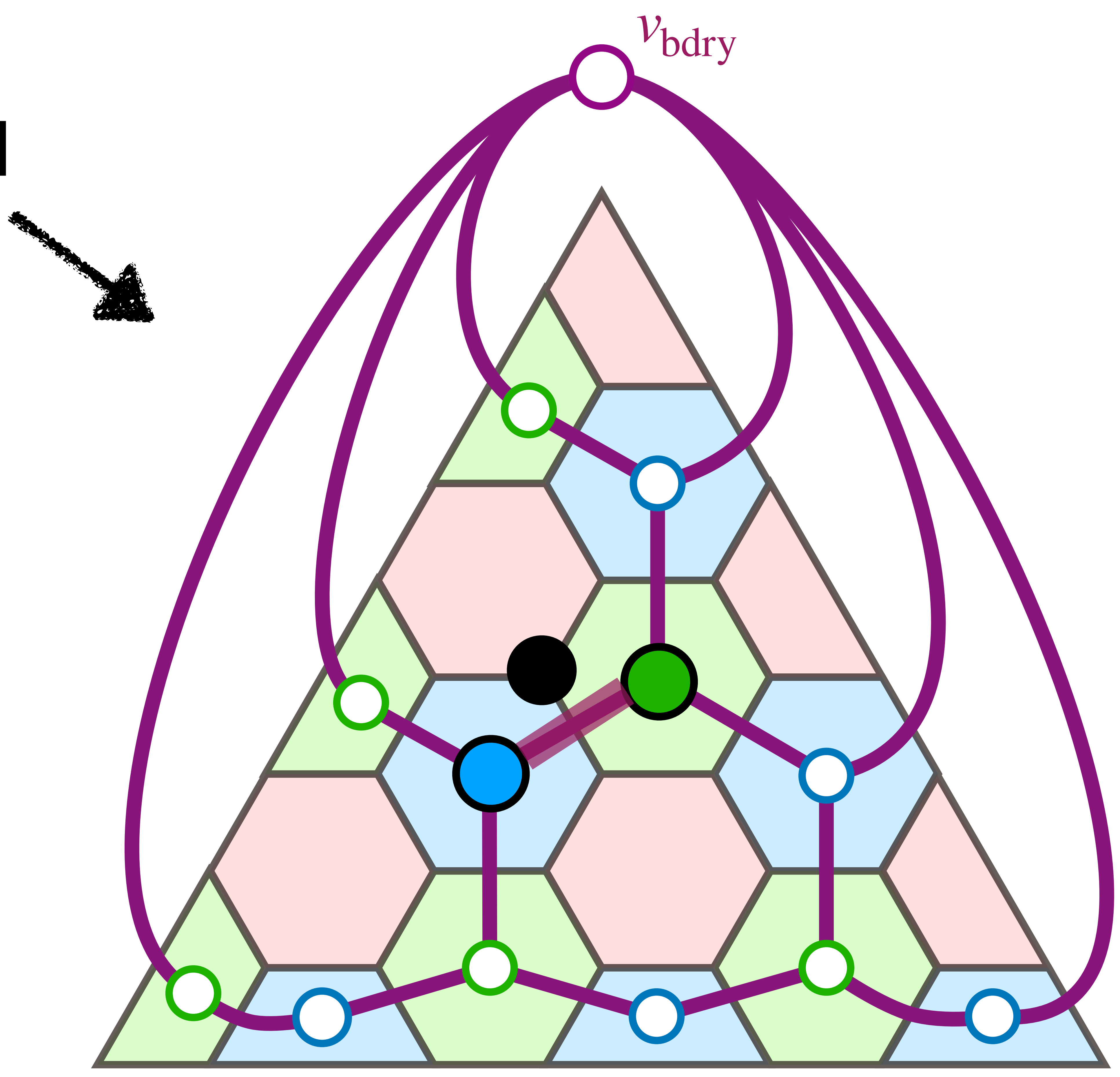
Elementary errors are not edge-like!

An elementary error affects at most three checks.

Decoding Color Codes



MWPM



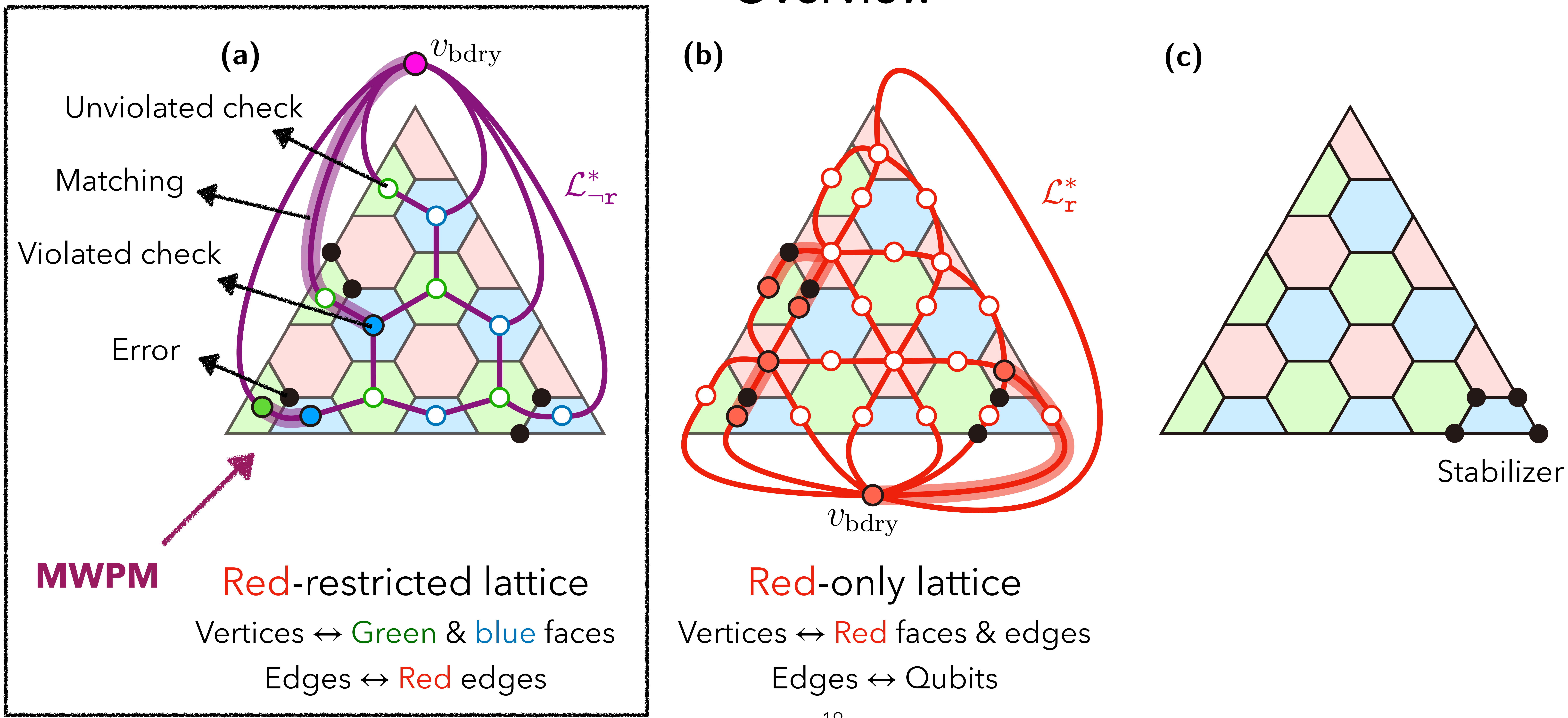
Red-restricted lattice

Vertices ↔ Green & blue faces

Edges ↔ Red edges

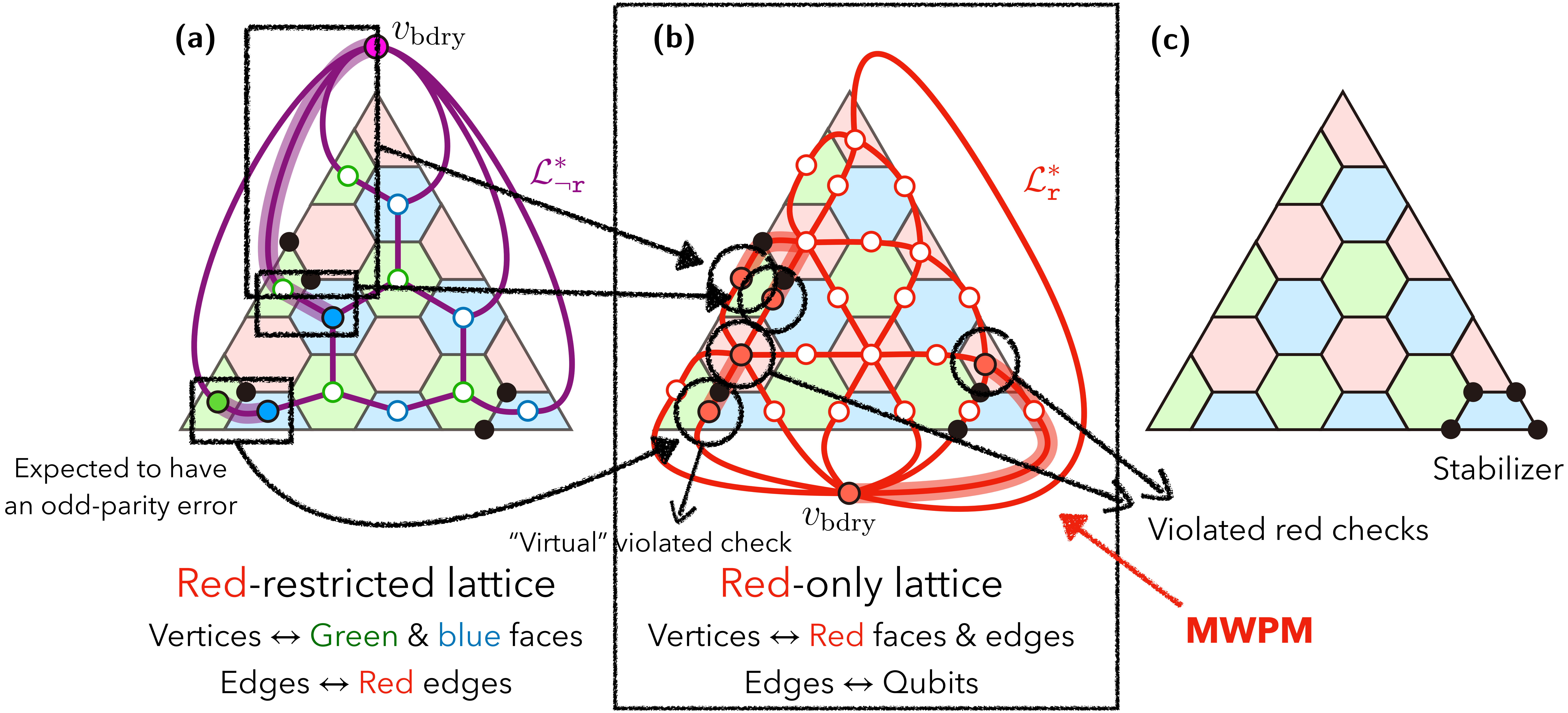
Concatenated MWPM Decoder

Overview



Concatenated MWPM Decoder

Overview



Red-restricted lattice

Vertices ↔ Green & blue faces

Edges ↔ Red edges

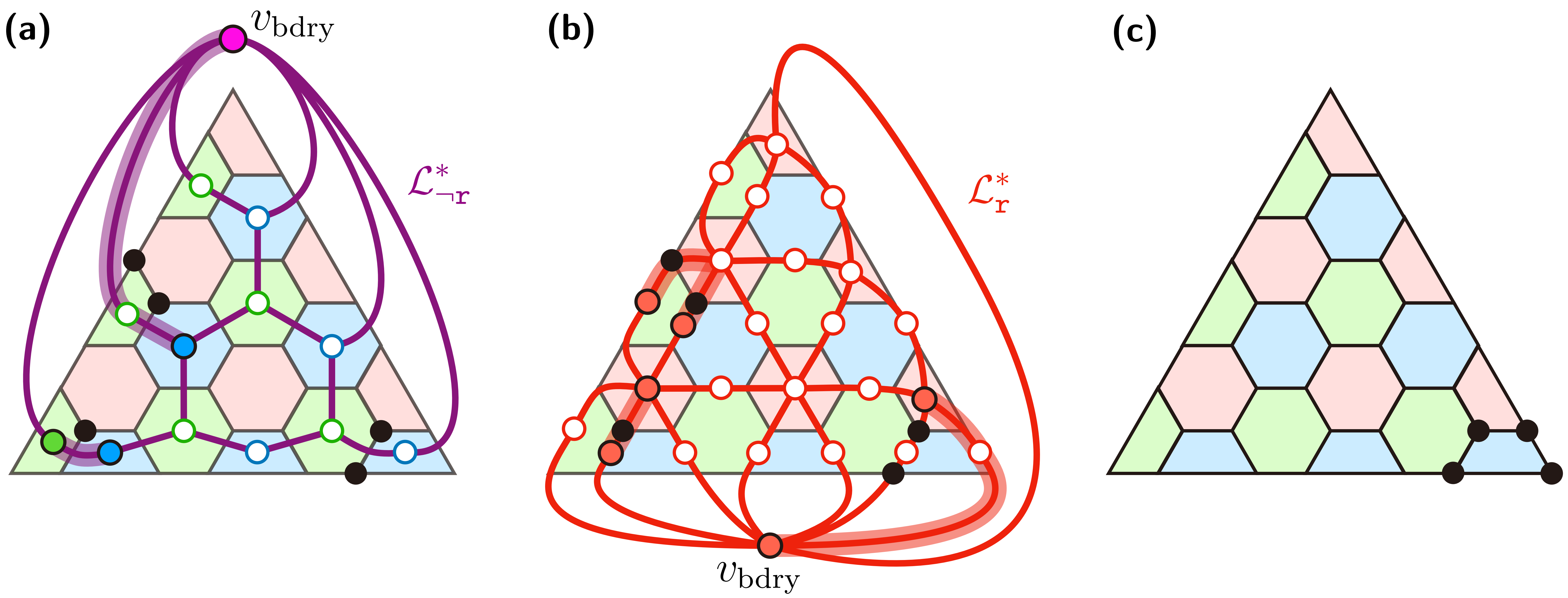
Red-only lattice

Vertices ↔ Red faces & edges

Edges ↔ Qubits

Concatenated MWPM Decoder

Overview



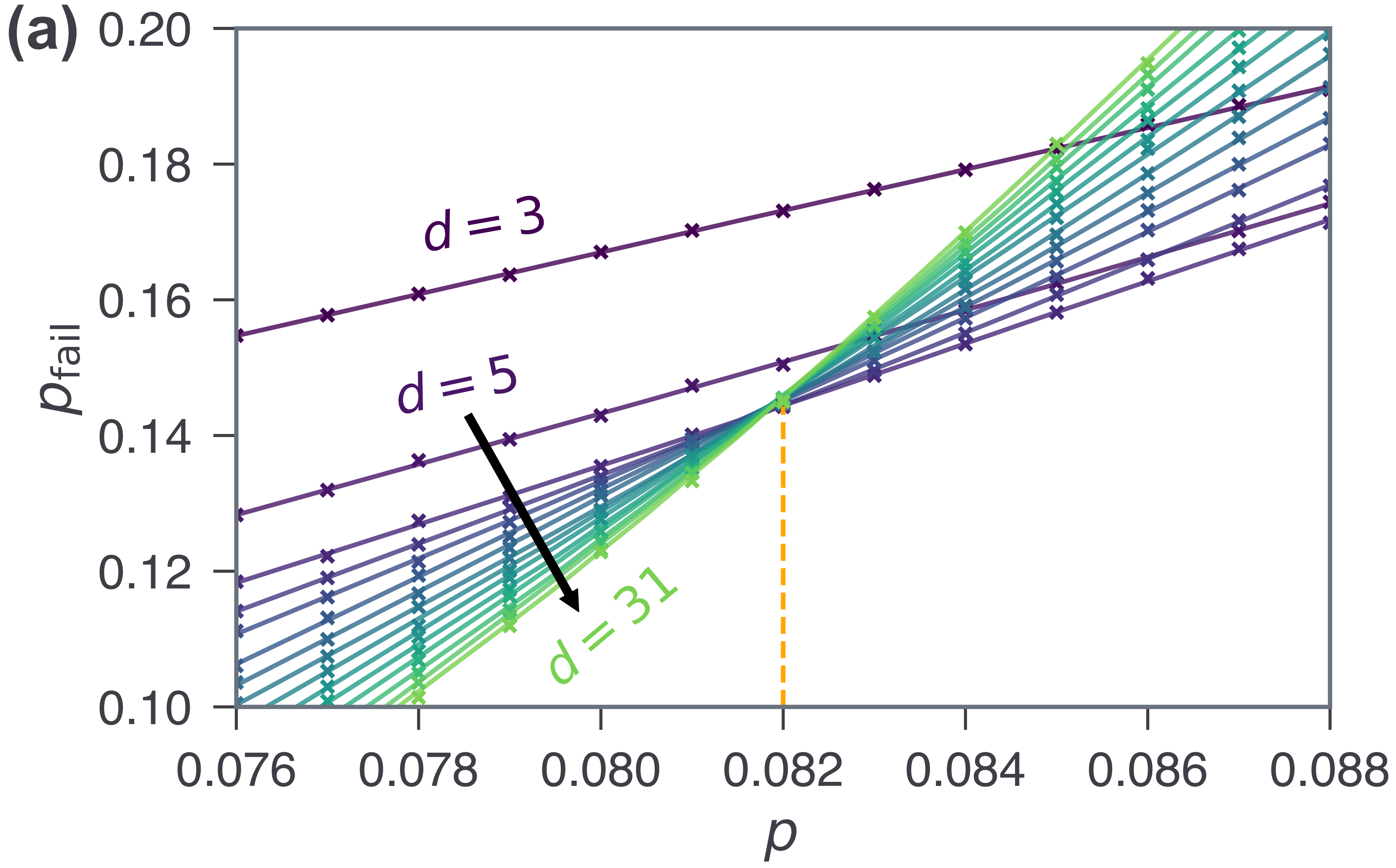
$\times\, 3$ by varying color: red, green, blue

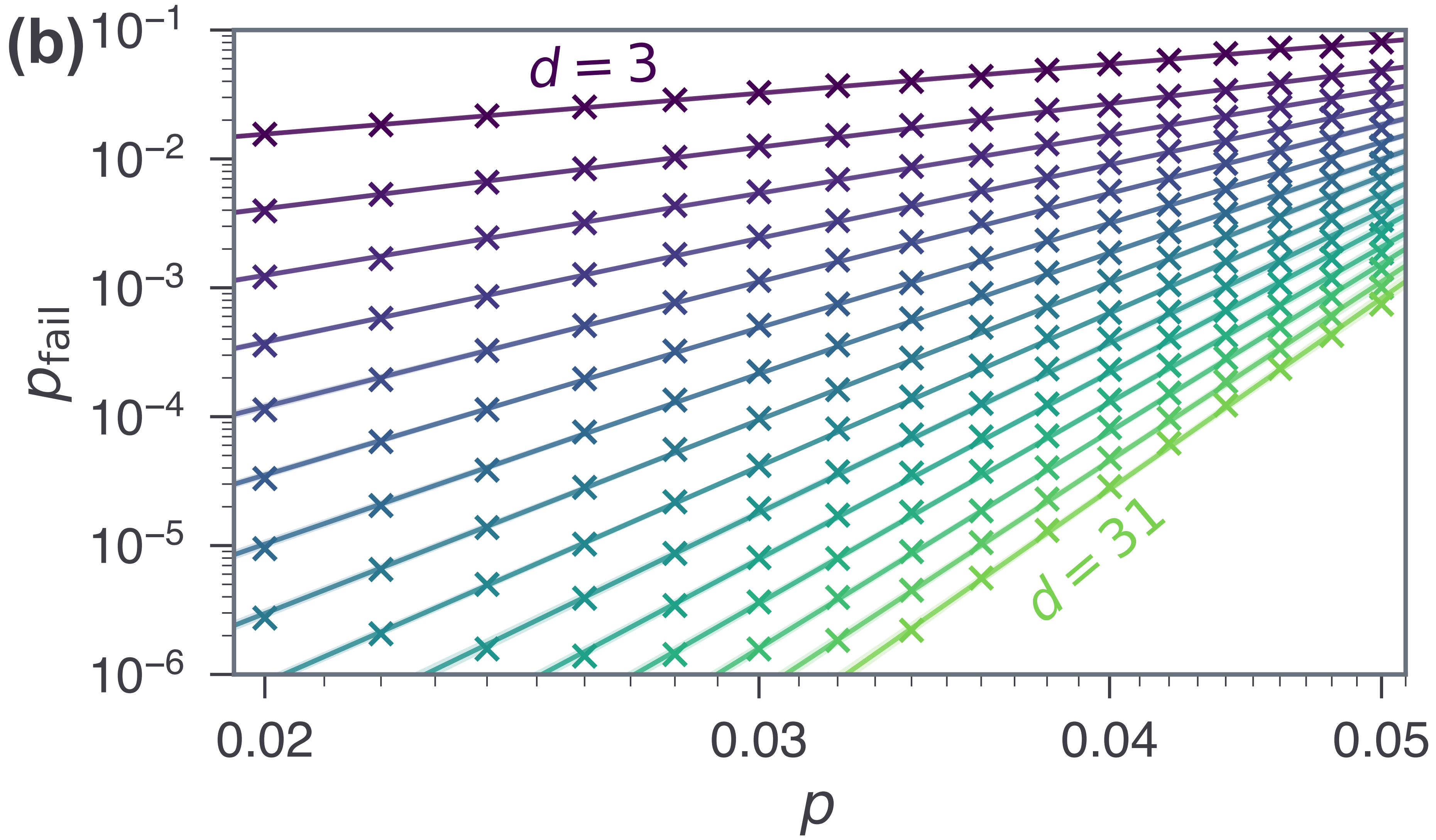
→ Select the smallest-weight correction

Concatenated MWPM Decoder

Performance

- Bit-flip noise model
  - Every data qubit undergoes a bit-flip (X) error with probability p.
  - Perfect syndrome measurements.





Noise threshold: $p_{\text{th}}^{\text{bitflip}} \approx 8.2\,\%$

($8.7\,\%$ for projection decoder and $9.0\,\%$ for Möbius decoder)

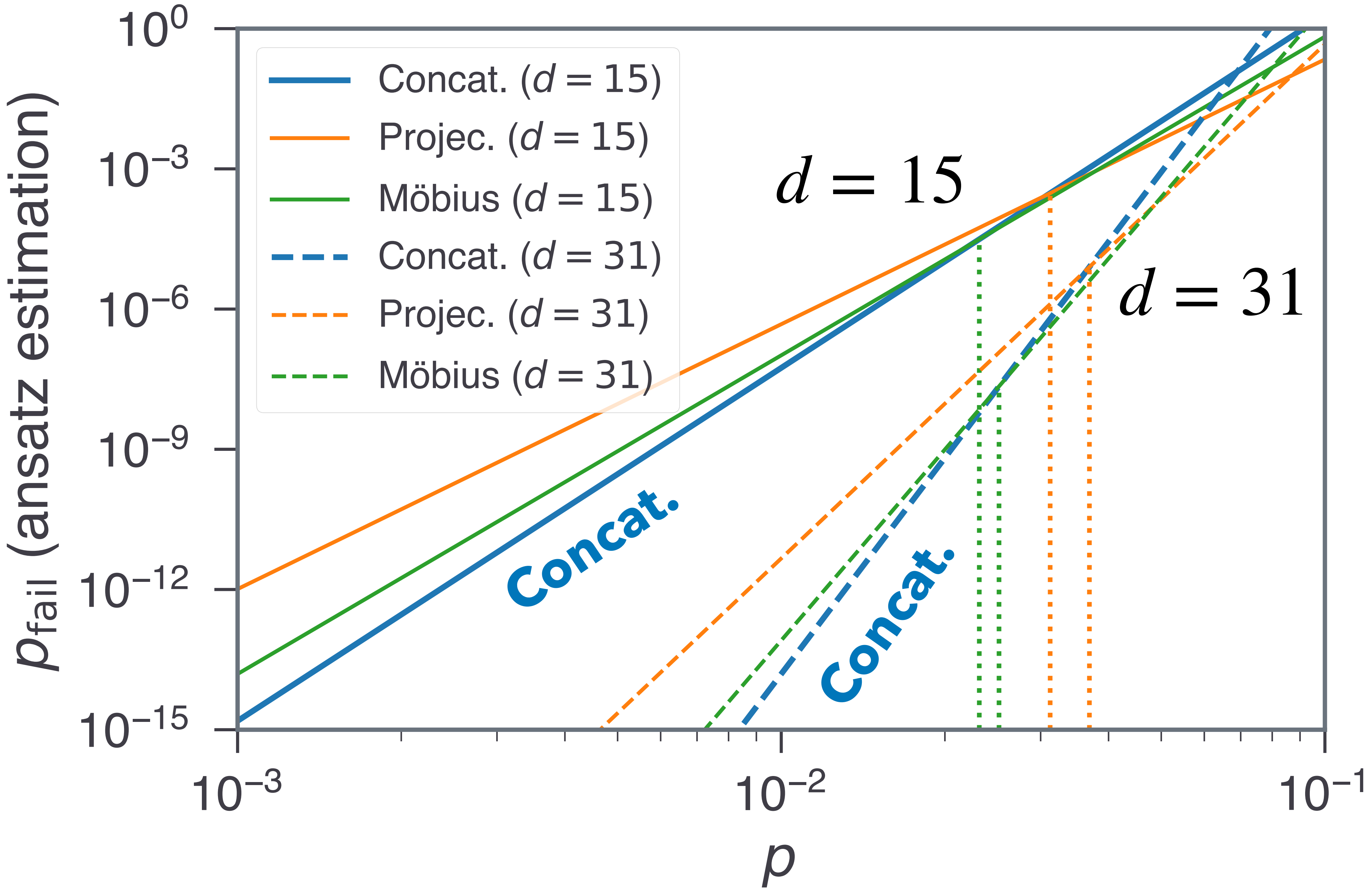
[Delfosse, PRA 2014] [Sahay & Brown, PRXQ 2022]

$$p_{\text{fail}} = \alpha \left(\frac{p}{p^{\star}}\right)^{\beta d + \eta} \approx 0.12 \left(\frac{p}{0.069}\right)^{0.49d + 0.17}$$

$\beta \approx 0.5$

Concatenated MWPM Decoder

Performance Comparison



$\beta \approx 1/3$ for the projection decoder

$\beta \approx 3/7$ for the Möbius decoder

(Due to low-weight uncorrectable errors)

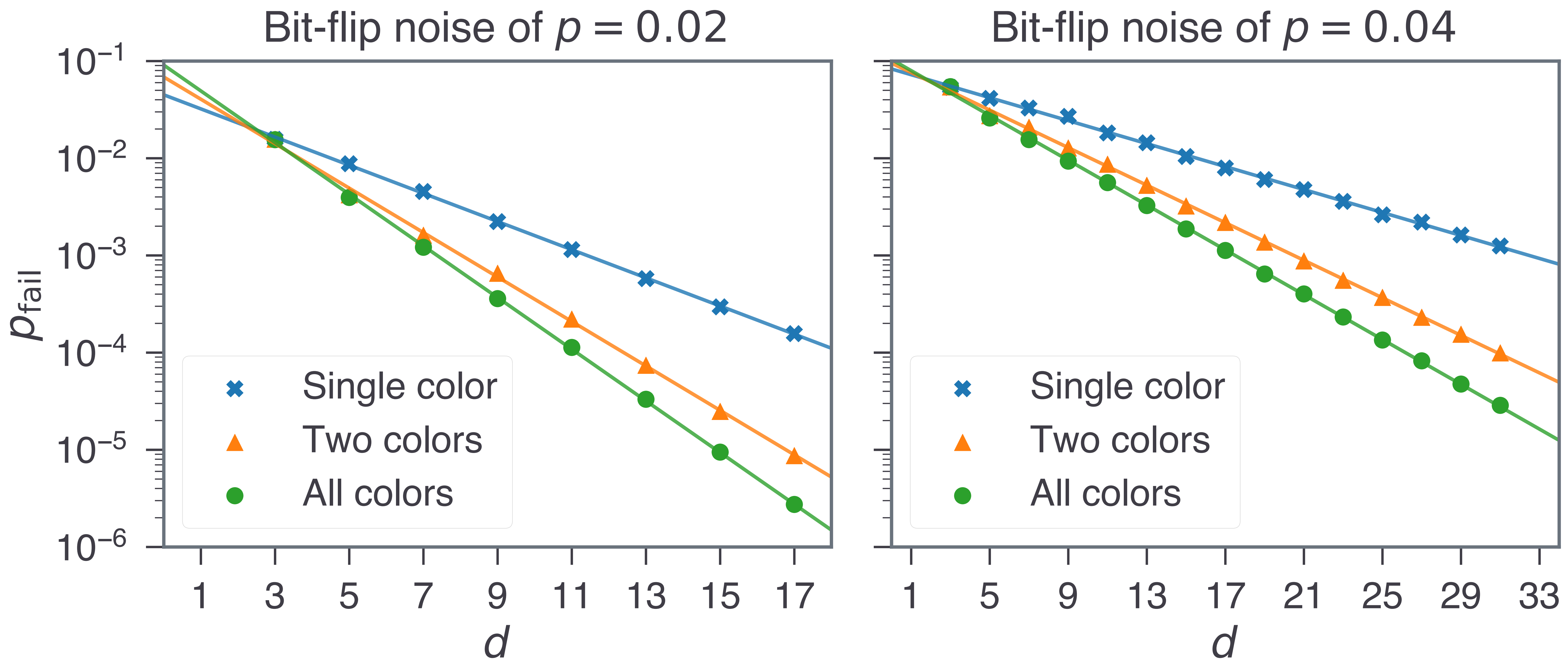
$w < d/2$

$$p_{\mathrm{fail}} = \alpha \left(\frac{p}{p^\star}\right)^{\beta d + \eta} \approx 0.12 \left(\frac{p}{0.069}\right)^{0.49d+0.17}$$

$\beta \approx 0.5$

Concatenated MWPM Decoder

Color Selecting Strategy Comparison



Generalization to Circuit-level Noise

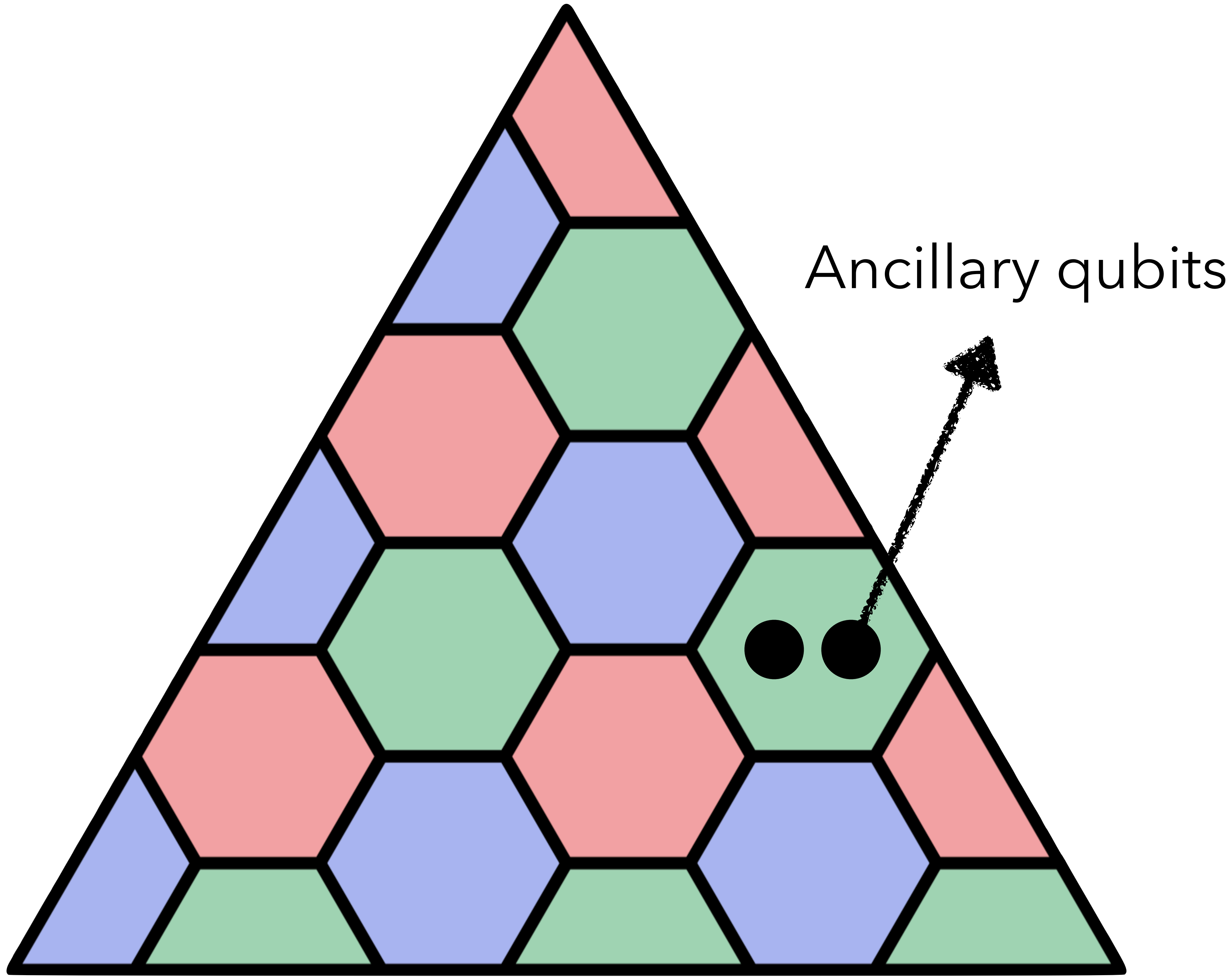
- We need to consider "circuits", not just codes.
- Circuit-level noise model
  - Every **measurement** outcome is flipped with probability p.
  - Every **preparation** of a qubit produces an orthogonal state with probability p.
  - Every single- or two-qubit **unitary gate** (including the idling gate I) is followed by a single- or two-qubit depolarizing noise channel with strength p.

$$\mathscr{E}_p^{(1)}\left(\rho_1\right) : \ \rho^{(1)} \mapsto (1-p)\rho^{(1)} + \frac{p}{3} \sum_{P \in \{X, Y, Z\}} P\rho^{(1)}P,$$

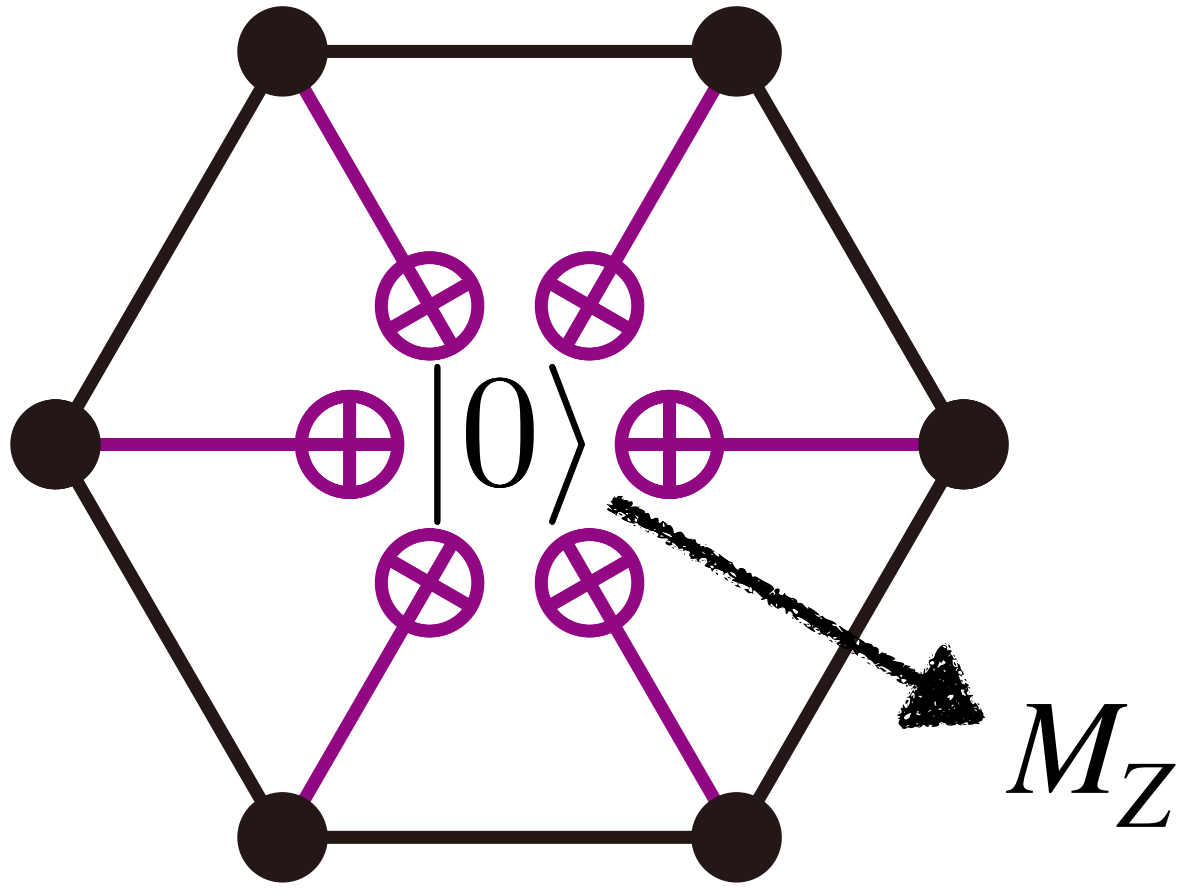
$$\mathscr{E}_p^{(2)}\left(\rho_2\right) : \ \rho^{(2)} \mapsto (1-p)\rho^{(2)} + \frac{p}{15} \sum_{\substack{P_1, P_2 \in \{I, X, Y, Z\} \\ P_1 \otimes P_2 \neq I \otimes I}} \left(P_1 \otimes P_2\right)\rho^{(2)}\left(P_1 \otimes P_2\right)$$

Generalization to Circuit-level Noise

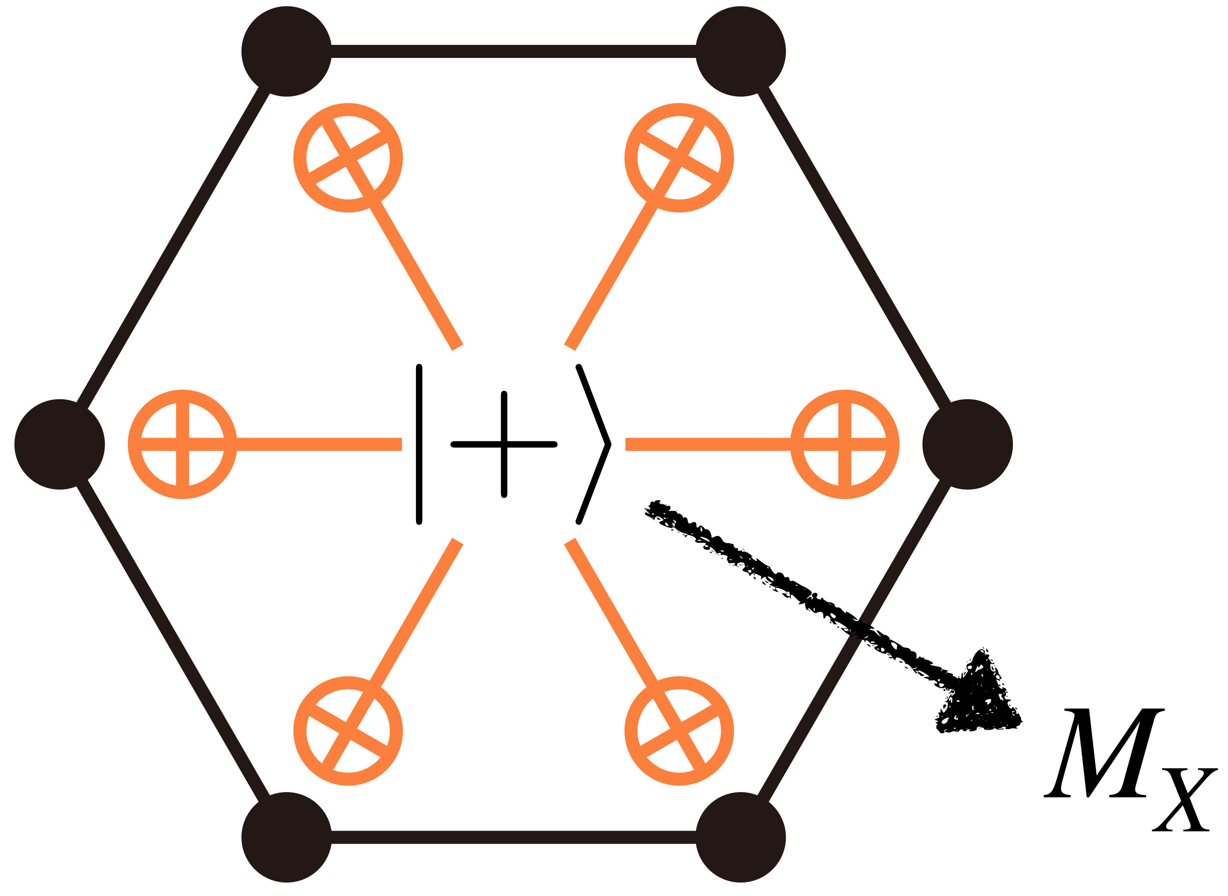
Syndrome Extraction



Z-type check measurement



X-type check measurement

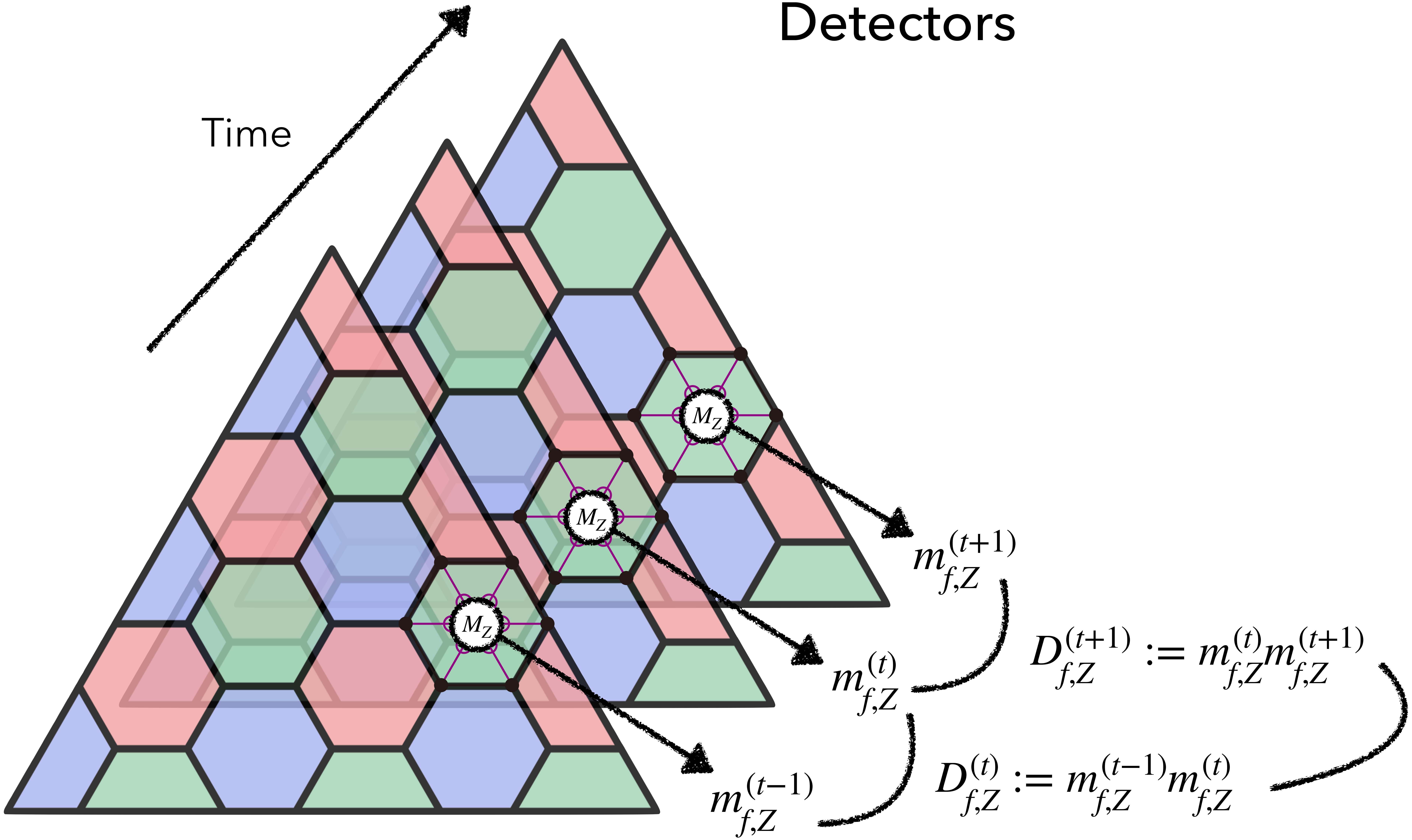


Order is important!

See [Beverland et al., PRXQ 2021]

Generalization to Circuit-level Noise

Detectors

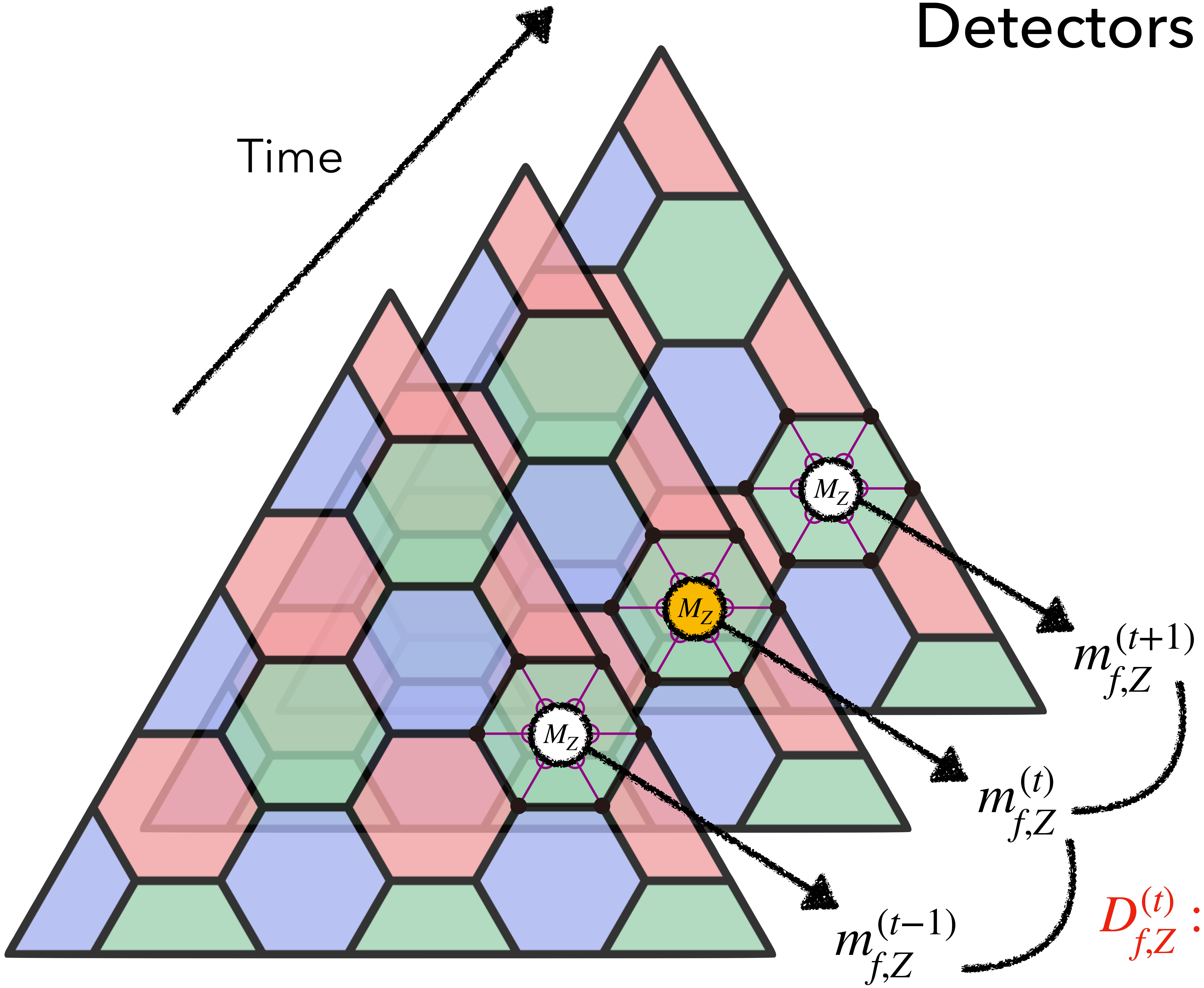


+1 if no errors

→ **Detectors**

Generalization to Circuit-level Noise

Detectors



$D_{f,Z}^{(t+1)} := m_{f,Z}^{(t)} m_{f,Z}^{(t+1)} = -1$ (Violated)

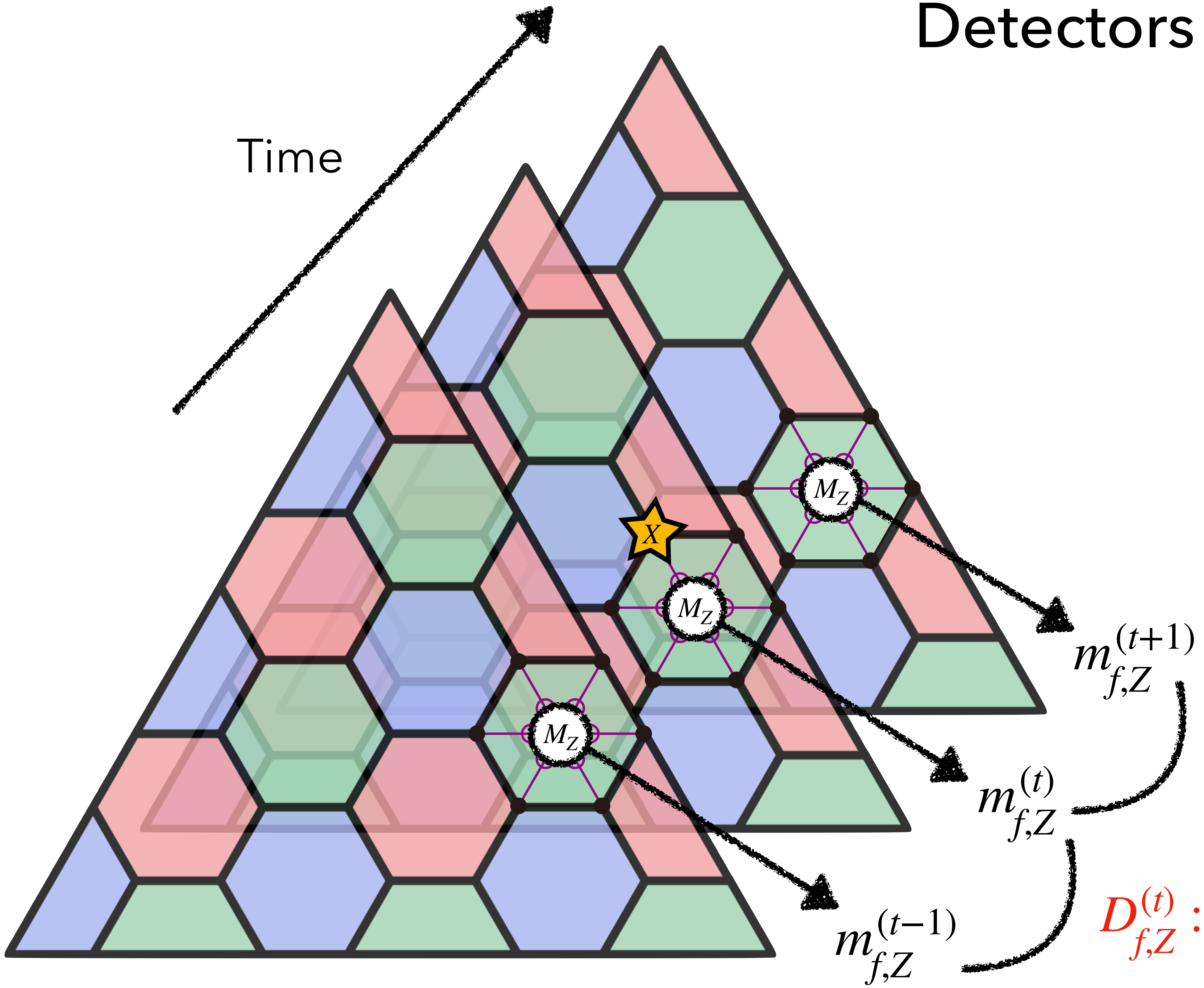
$D_{f,Z}^{(t)} := m_{f,Z}^{(t-1)} m_{f,Z}^{(t)} = -1$ (Violated)

$+1$ if no errors

→ **Detectors**

Generalization to Circuit-level Noise

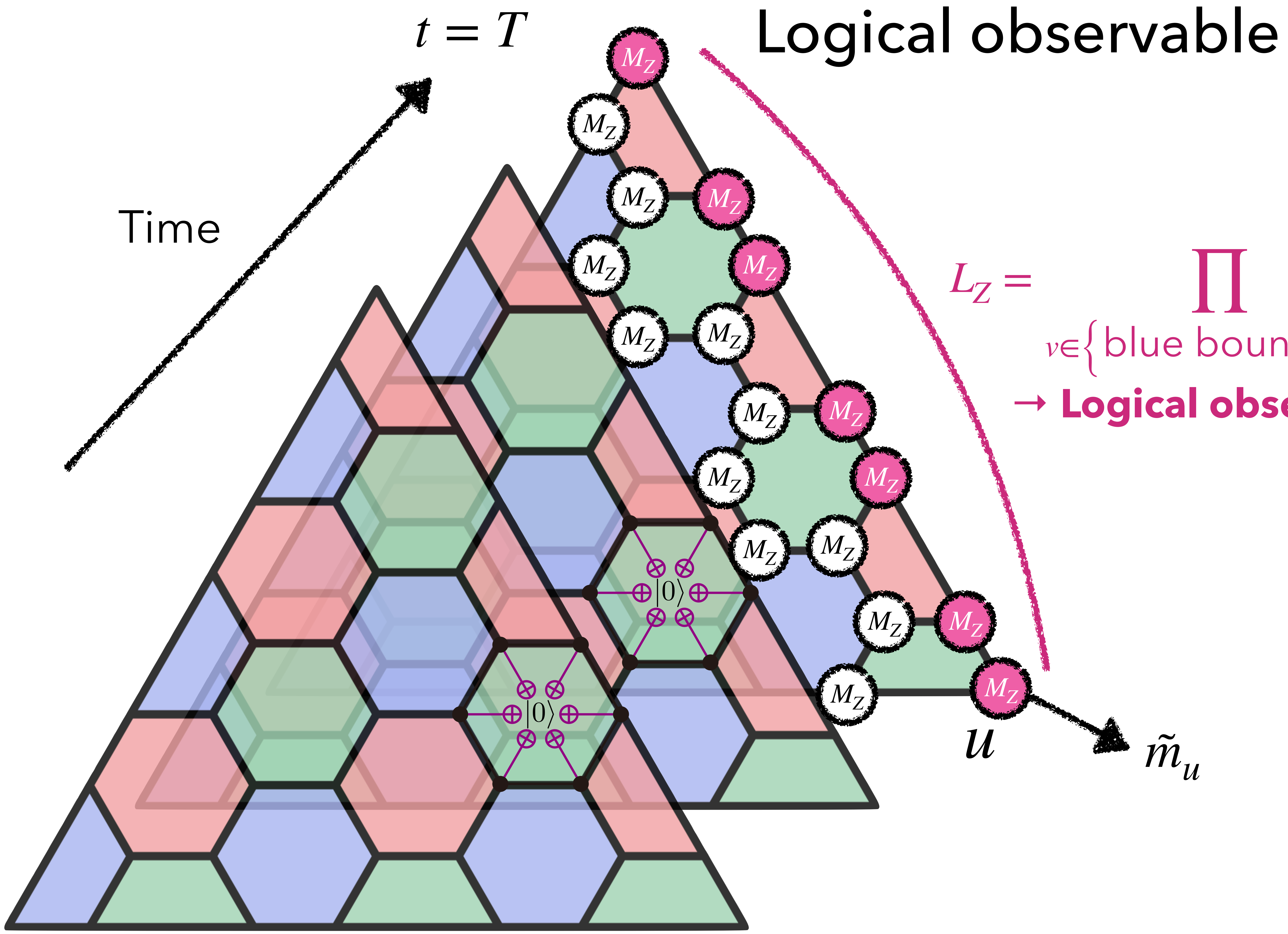
Detectors



$D_{f,Z}^{(t+1)} := m_{f,Z}^{(t)} m_{f,Z}^{(t+1)} = +1$

$D_{f,Z}^{(t)} := m_{f,Z}^{(t-1)} m_{f,Z}^{(t)} = -1$ (Violated)

Generalization to Circuit-level Noise



$$L_Z = \prod_{v \in \{\text{blue boundary}\}} \tilde{m}_v$$

→ **Logical observable**

Generalization to Circuit-level Noise

Detector Error Model (DEM)

- Definition
  - A set of independent **error mechanisms**.
  - Each error mechanism is specified by
    - its **probability**,
    - **detectors** flipped by it,
    - **logical observables** flipped by it.
- Construction
  - Decompose depolarizing noise channels as
    $\mathscr{E}_p^{(1)} = \mathscr{E}_{q_1}^{(X)} \circ \mathscr{E}_{q_1}^{(Y)} \circ \mathscr{E}_{q_1}^{(Z)}, \quad \mathscr{E}_p^{(2)} = \mathscr{E}_{q_2}^{(X\otimes I)} \circ \mathscr{E}_{q_2}^{(X\otimes X)} \circ \mathscr{E}_{q_2}^{(X\otimes Y)} \circ \cdots \circ \mathscr{E}_{q_2}^{(Z\otimes Z)},$
    where $q_1 := (1-\sqrt{1-4p/3})/2$ and $q_2 := [1-(1-16p/15)^{1/8}]/2$

    $\mathscr{E}_q^{(P)}(\rho) := (1-q)\rho + qP\rho P$
  - Commute all the single-Pauli error channels to the end of the circuit.
  - Check detectors & logical observables flipped by each single-Pauli error channel
- ex) $d=3, T=1, p=10^{-3}$ →

Using *Stim* library [Gidney, Quantum 2021]

```
stim.DetectorErrorModel('''
    error(0.00193118) D0 D1 D2
    error(0.0012662) D0 D1 L0
    error(0.00259527) D0 D2
    error(0.00504495) D0 D3
    error(0.00458224) D0 L0
    error(0.00119929) D1
    error(0.0012662) D1 D2
    error(0.000799787) D1 D2 D3
    error(0.000799787) D1 D3 D5
    error(0.00266116) D1 D3 L0
    error(0.00504495) D1 D4
    error(0.00133227) D1 D5
    error(0.00325848) D1 L0
    error(0.00193118) D2
    error(0.000799787) D2 D3
    error(0.00504495) D2 D5
    error(0.000533333) D2 L0
    error(0.00272788) D3 D4 D5
    error(0.00286063) D3 D4 L0
    error(0.00339091) D3 D5
    error(0.00405306) D3 L0
    error(0.00471432) D4 D5
    error(0.00352348) D4 L0
    error(0.00484654) D5
    error(0.000533333) D5 L0
''')
```

Generalization to Circuit-level Noise

MWPM on DEM

- If a DEM only has **edge-like** error mechanisms (i.e., each error mechanism affects at most two detectors),

MWPM

DEM

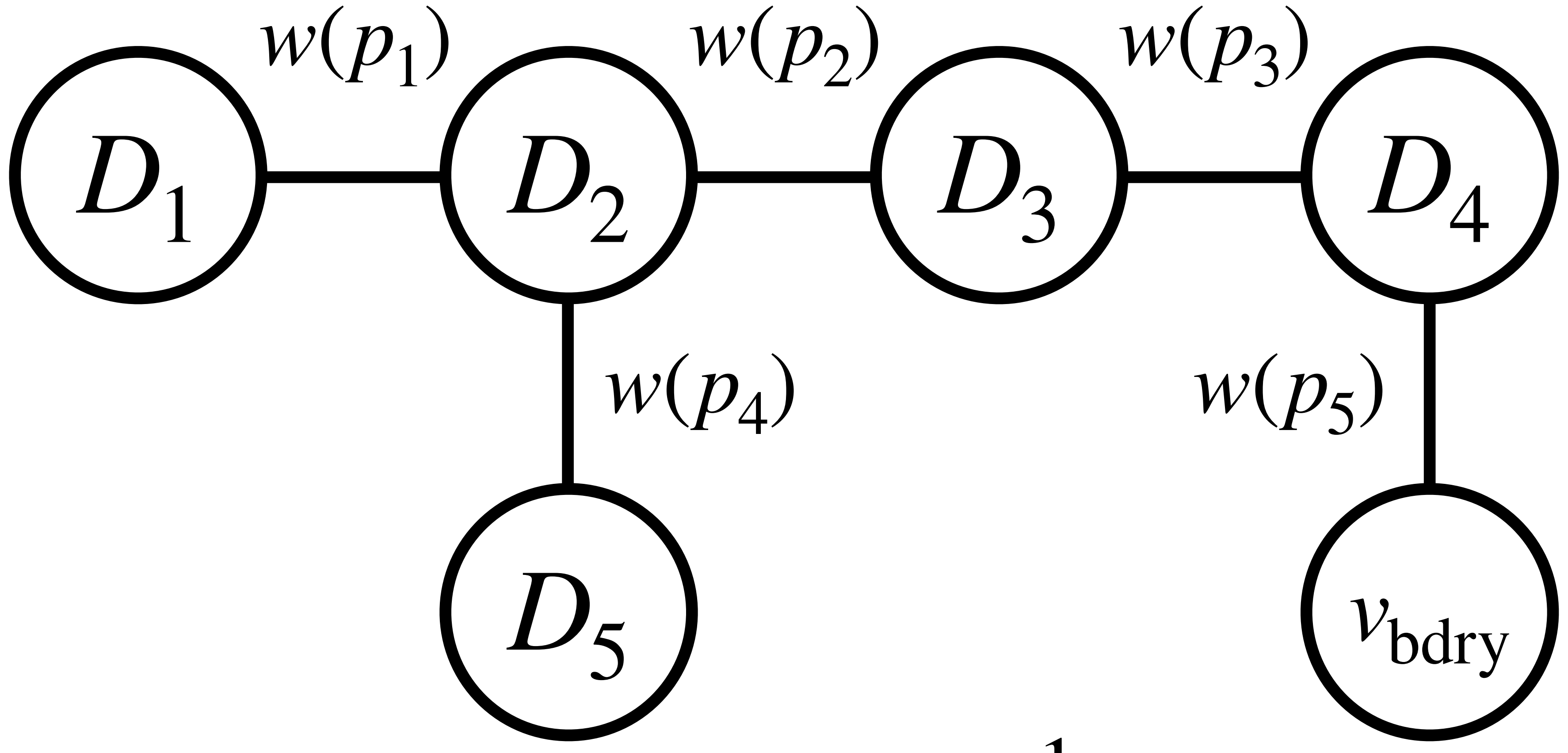
$p_1 \to D_1, D_2,$

$p_2 \to D_2, D_3,$

$p_3 \to D_3, D_4,$

$p_4 \to D_2, D_5,$

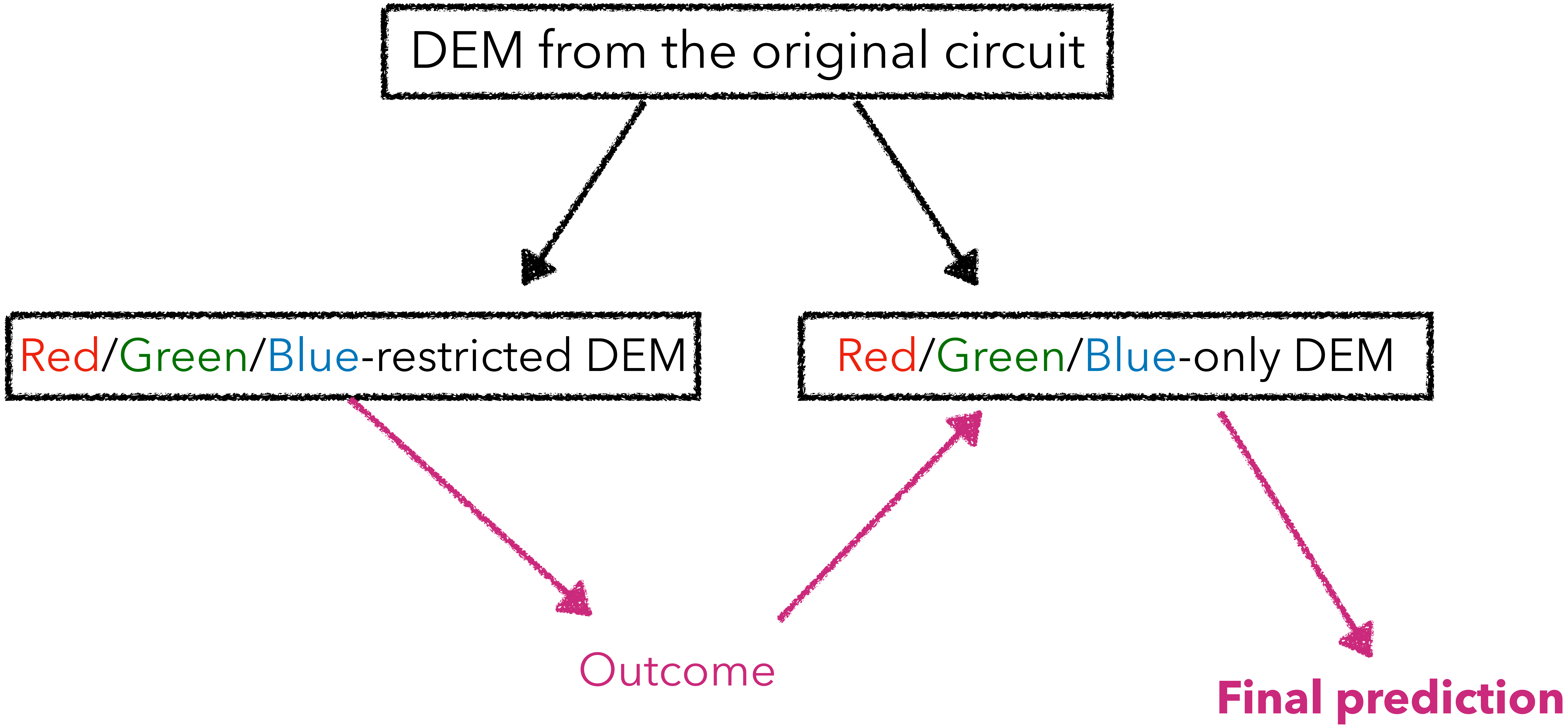
$p_5 \to D_4$



$$w(p) := \log \frac{1-p}{p}$$

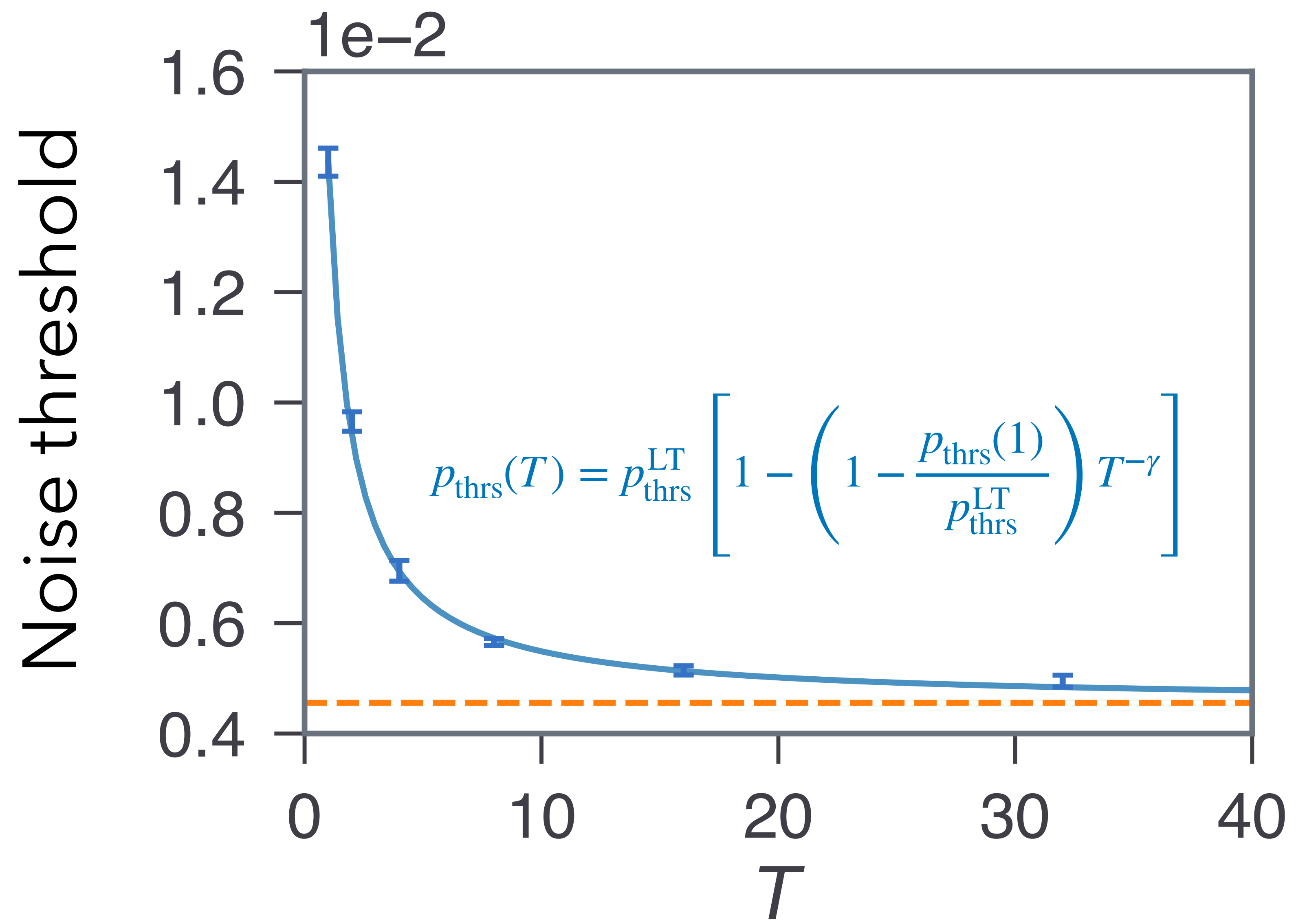
Generalization to Circuit-level Noise

Generalized Concatenated MWPM decoder



Generalization to Circuit-level Noise

Performance



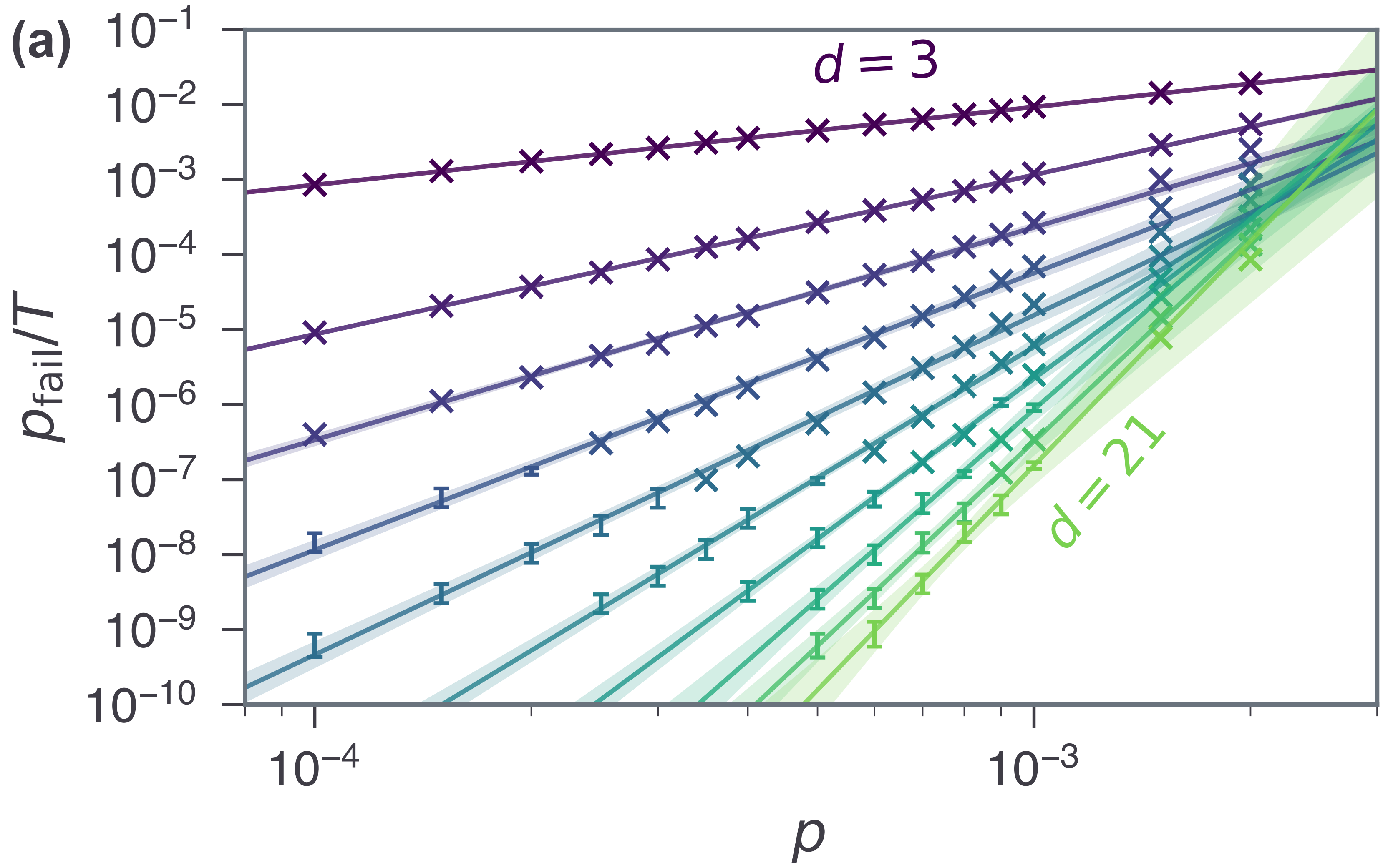
$$p_{\rm thrs}^{\rm LT} := \lim_{T\to\infty} p_{\rm thrs}(T) \approx 0.456\,\%$$

(Projection decoder)

$$p_{\rm thrs}^{\rm LT} = \begin{cases} 0.37\,\% & \text{in [Beverland et al., PRXQ 2021],} \\ 0.47\,\% & \text{in [Zhang et al., arXiv:2309.05222]} \end{cases}$$

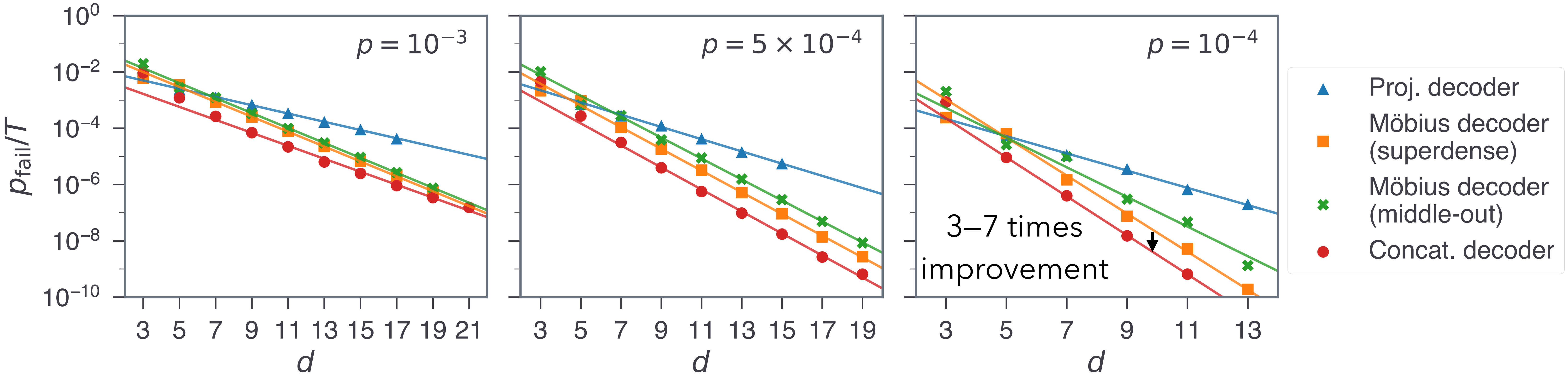
(Möbius decoder)

$0.5\,\% \lesssim p_{\rm thrs}^{\rm LT} \lesssim 0.7\,\%$ [Gidney & Jones, arXiv:2312.08813]



$$p_{\rm fail} \approx 0.0091 \left(\frac{p}{0.0032}\right)^{0.51d-0.66}$$

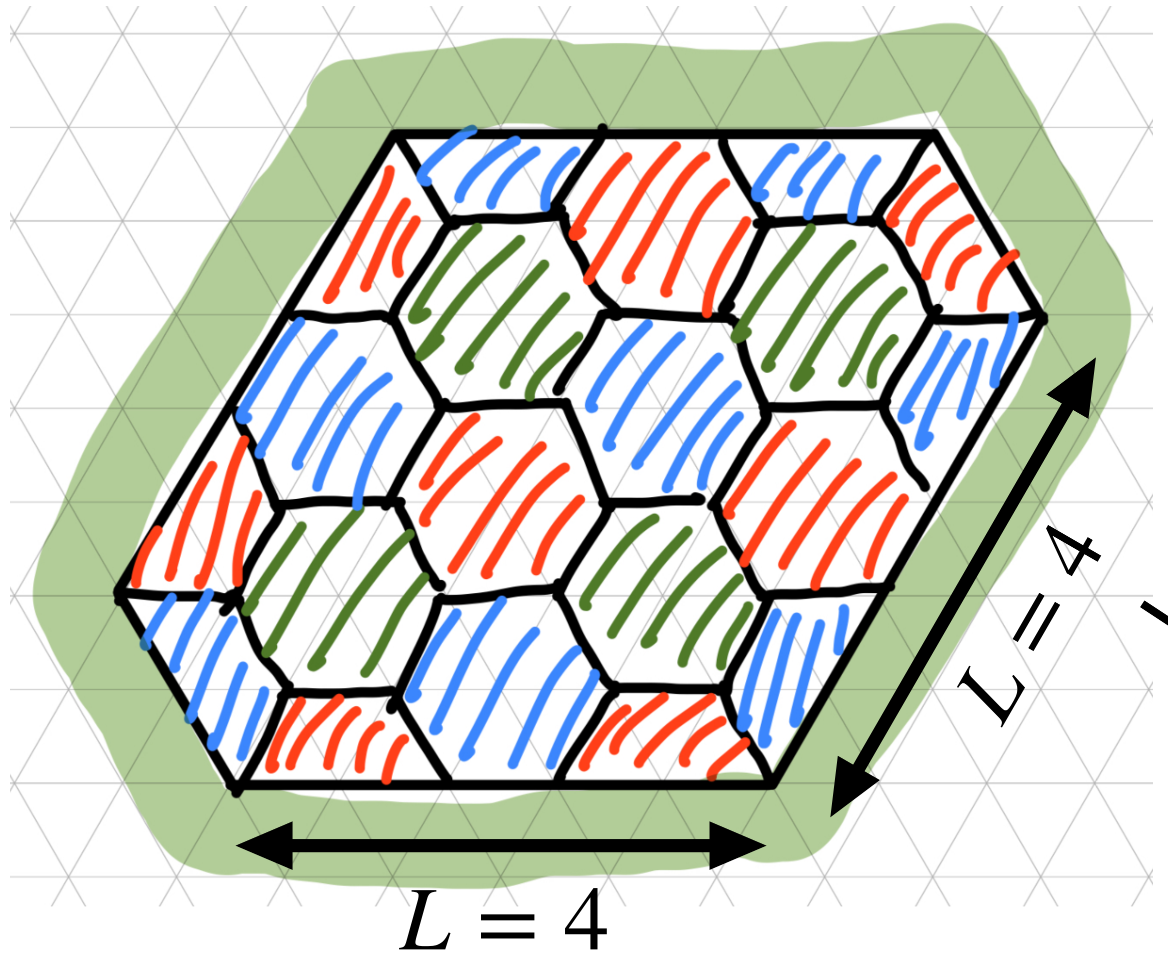
β



Data on projection decoder from [Beverland et al., PRXQ 2021] & Data on Möbius decoder from [Gidney & Jones, arXiv:2312.08813]

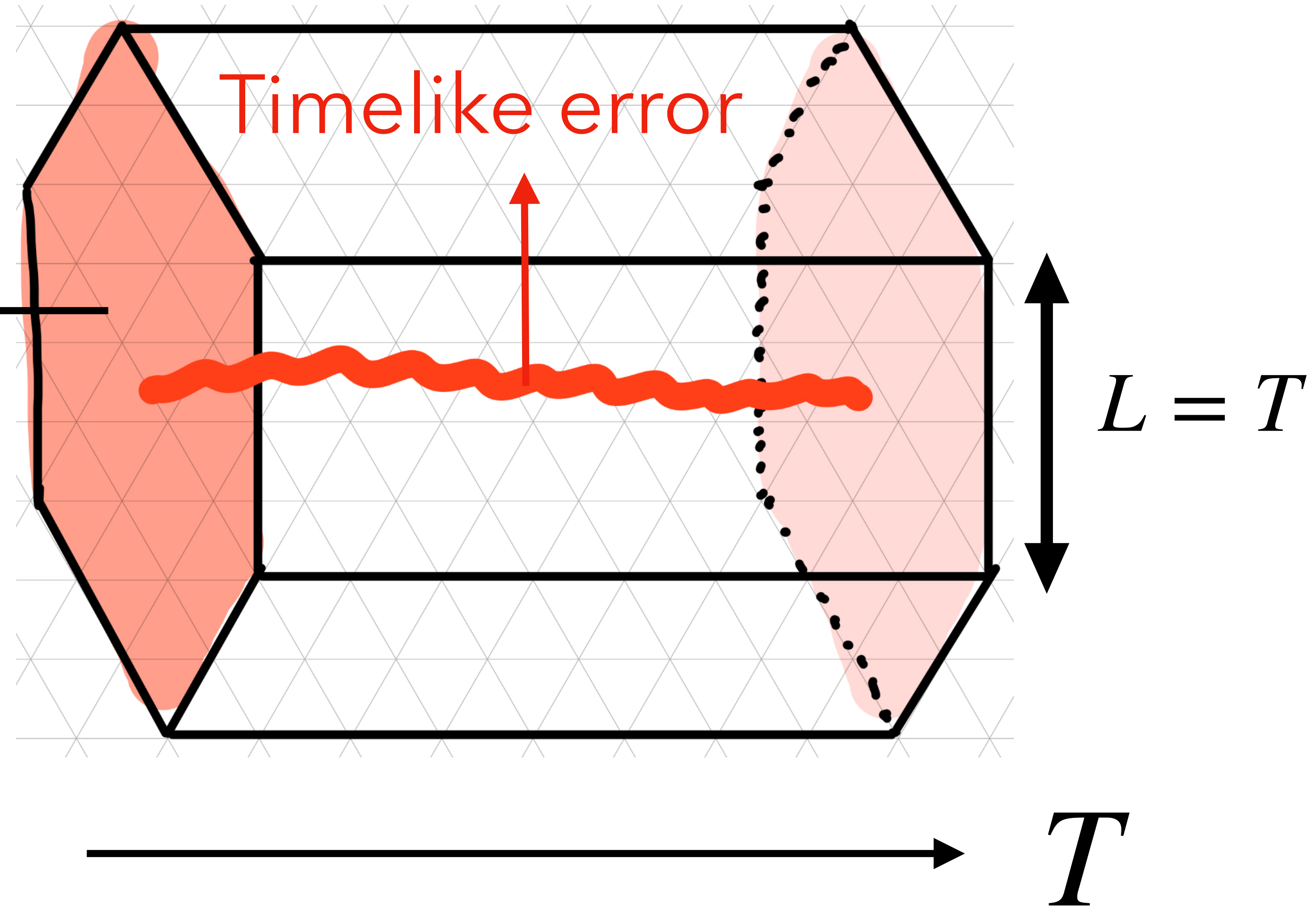
Generalization to Circuit-level Noise

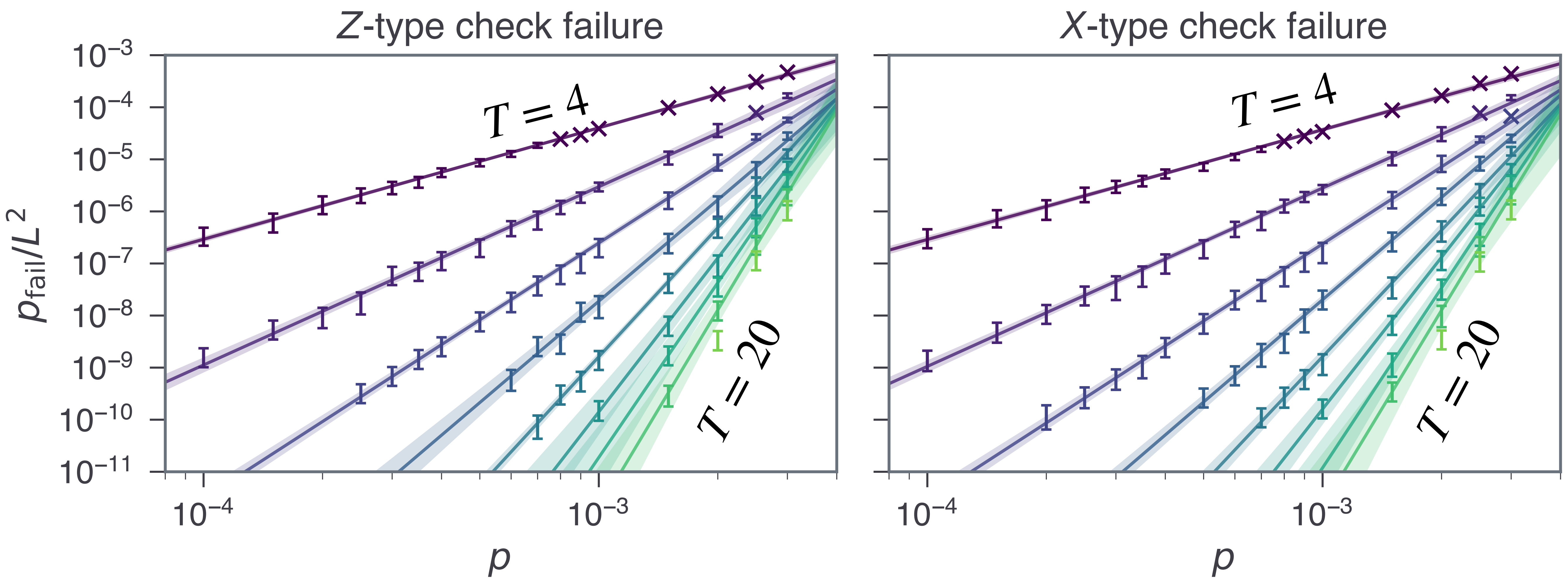
Stability experiments (timelike errors)



Red temporal boundary
(Bell measurements on red edges)

Encode no logical qubits





Z-type check failure

$$p_{\text{fail}}/L^2 = \left(7.1 \times 10^{-4}\right) \times \left(\frac{p}{0.0048}\right)^{0.77T-1.1}$$

> 0.5?

X-type check failure

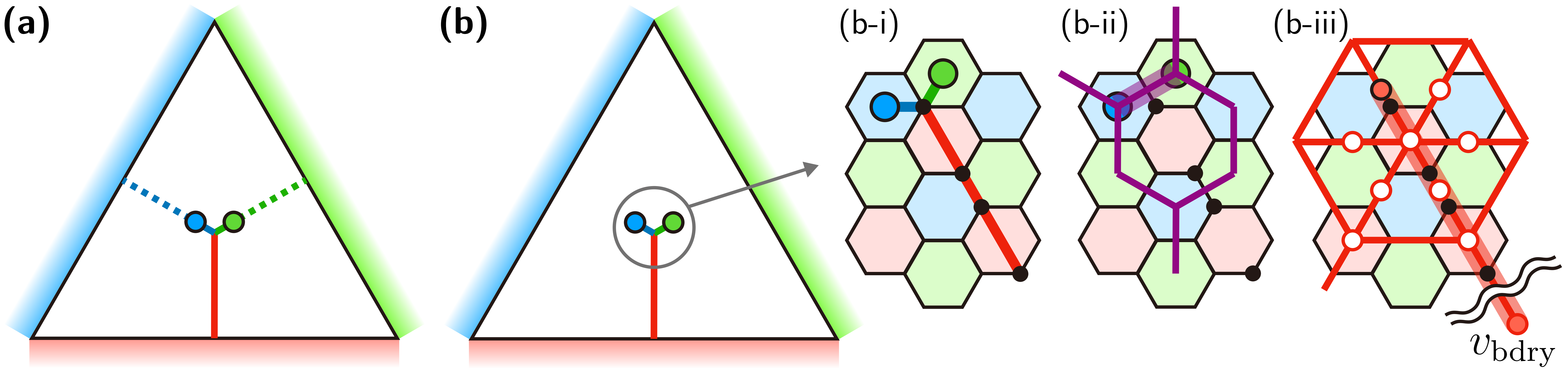
$$p_{\text{fail}}/L^2 = \left(6.8 \times 10^{-4}\right) \times \left(\frac{p}{0.0048}\right)^{0.77T-1.1}$$

Summary

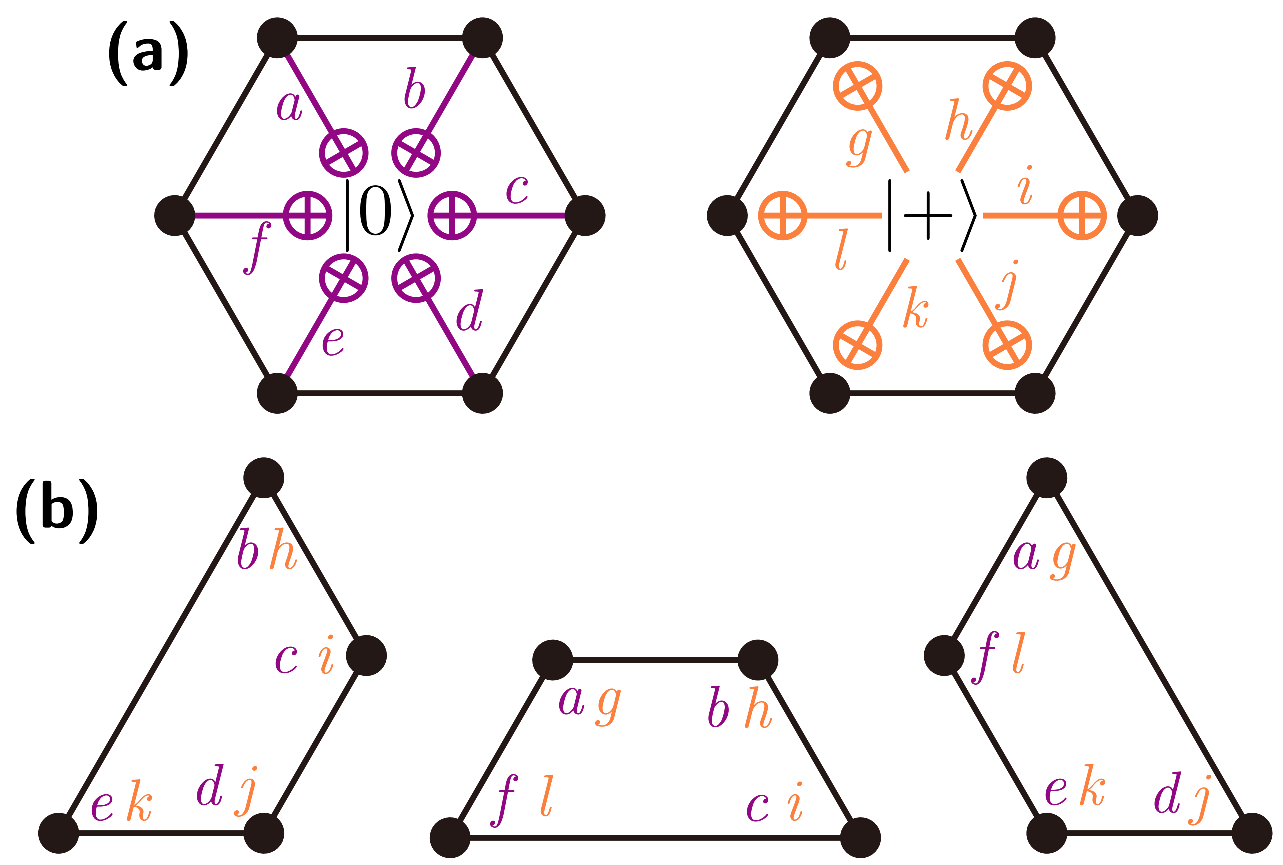
- Color codes have many advantages such as low resource costs and transversal Clifford gates, but their decoding is relatively difficult.
- To apply MWPM, elementary errors (or error mechanisms in the detector error model) should be edge-like, which is not the case for color codes.
- The concatenated MWPM decoder resolves this problem by using the concatenation of two MWPMs on two lattices per color.
- The decoder can be generalized to accommodate circuit-level noise by using detector error models.
- Its sub-threshold scaling nearly achieves $p_{\mathrm{fail}} \sim p^{d/2}$ and outperforms previous decoders.

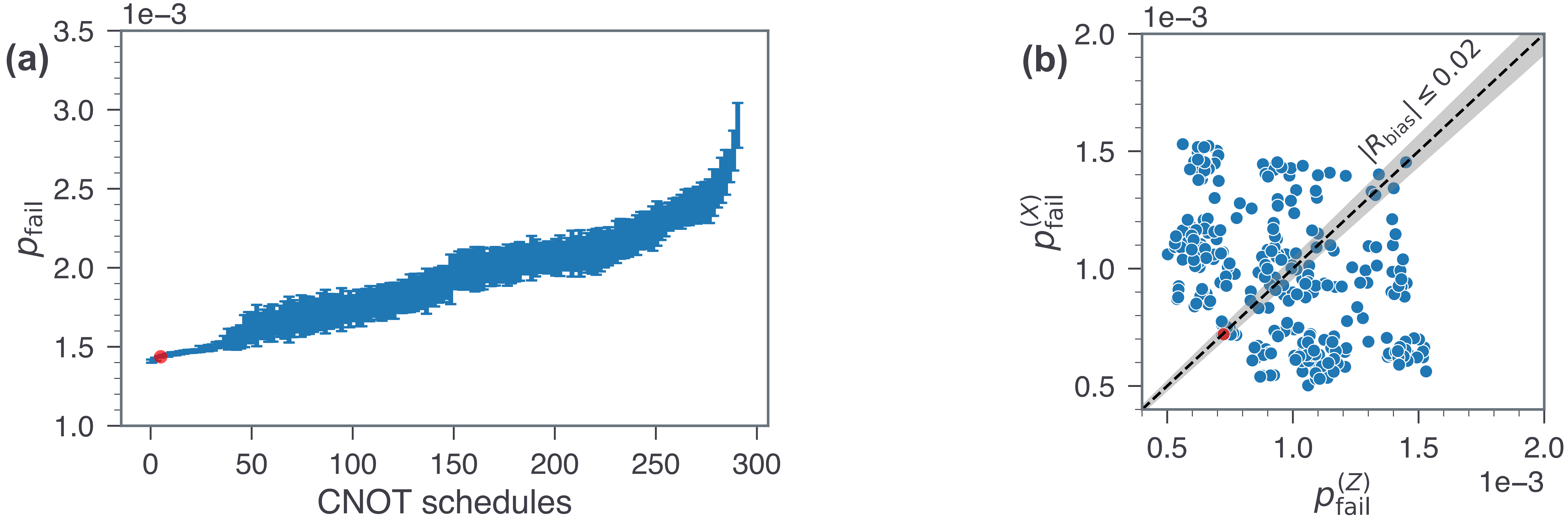
Thank you

Correctable O(d/3) error

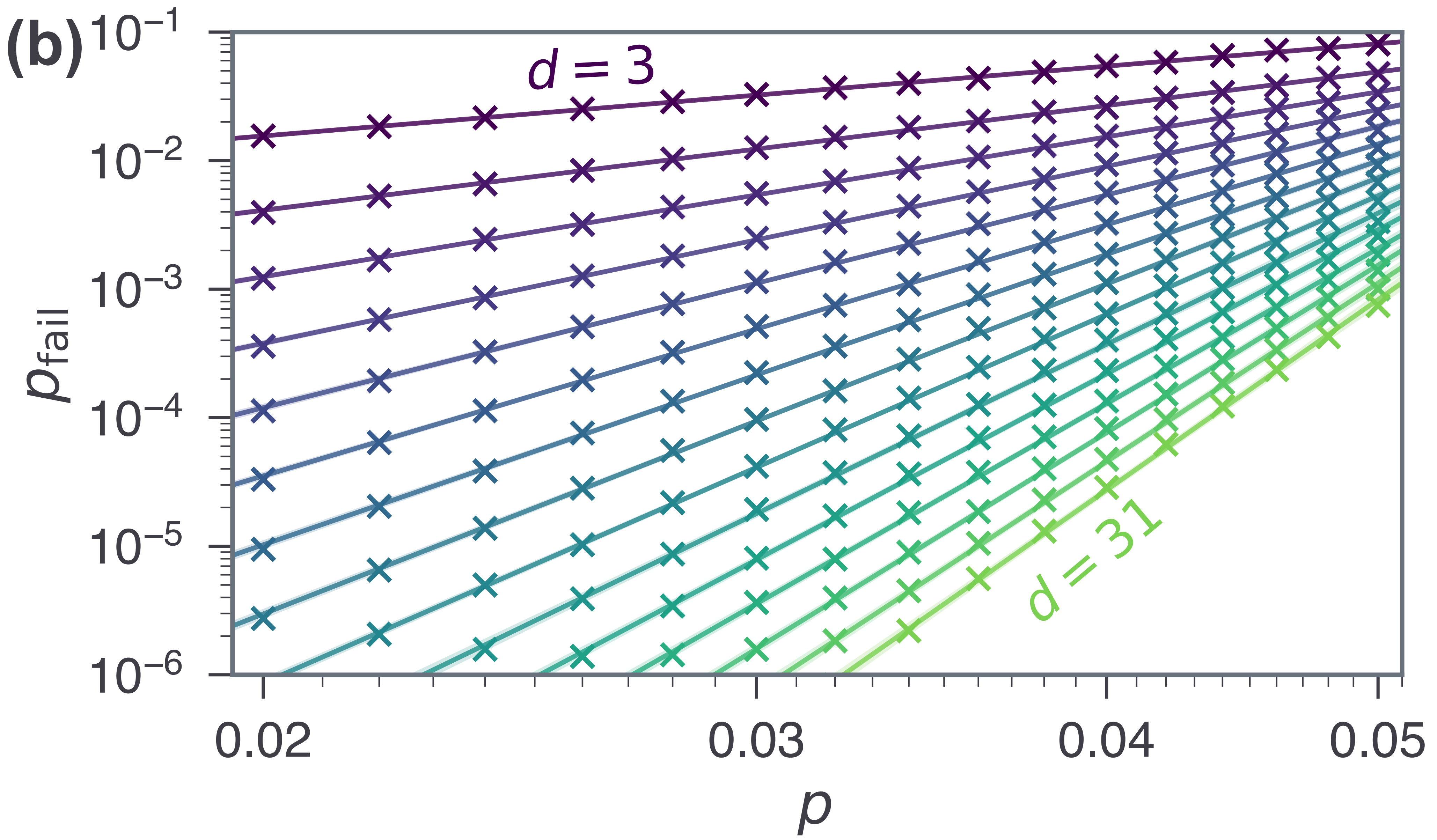


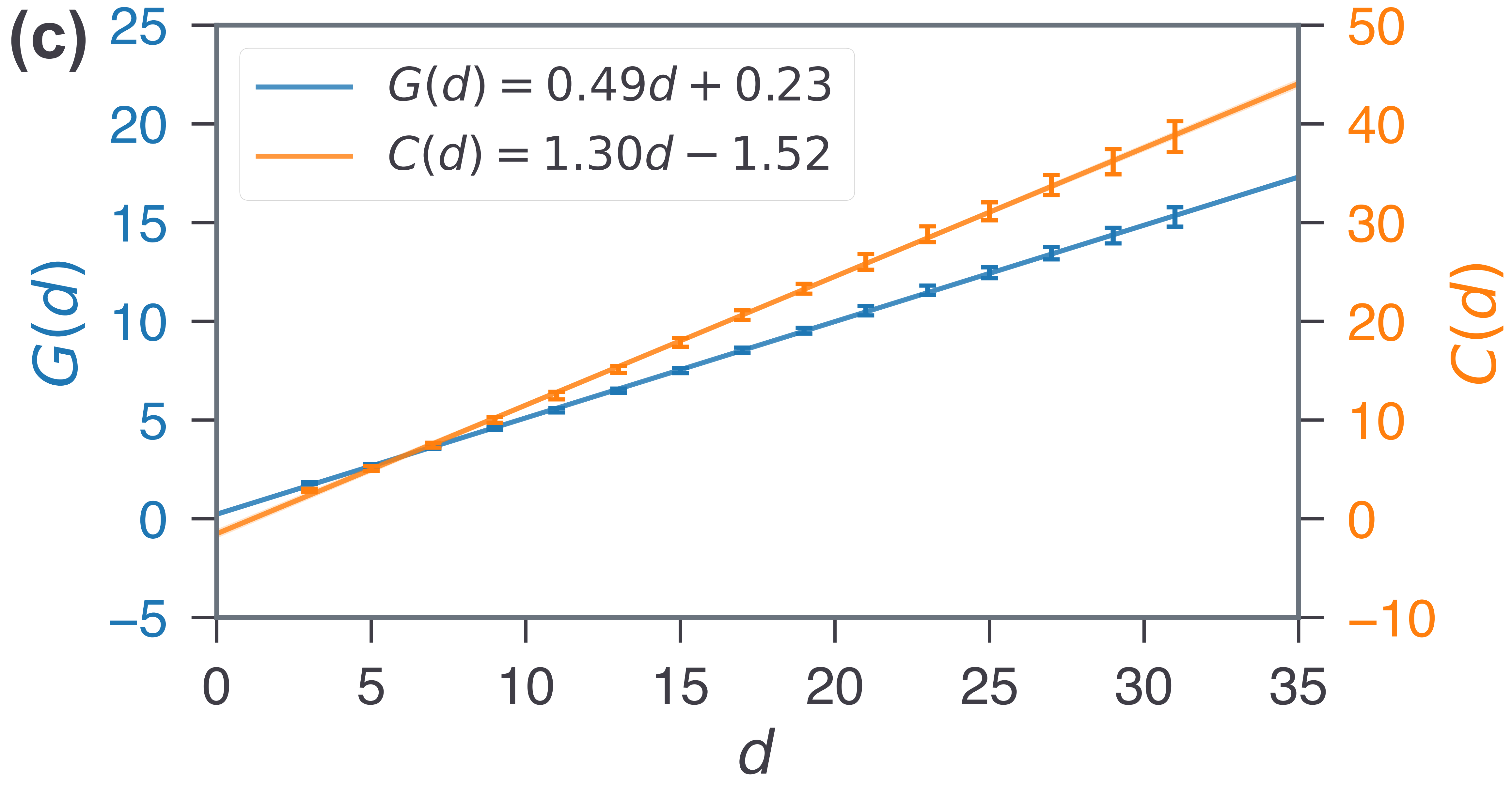
CNOT schedule comparison



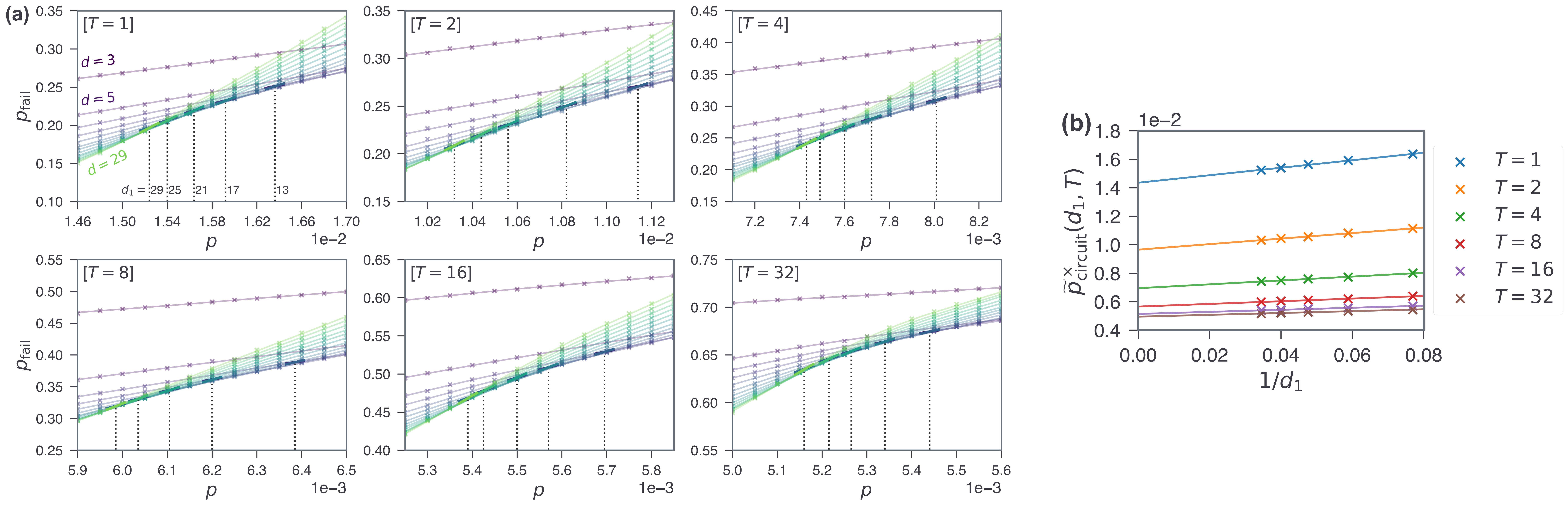


More simulation data: Bit-flip





More simulation data: Circuit-level



Color selection strategy comparison